

The Riemann Integral

Partition of a closed Interval

Let $[a, b]$ be a finite closed interval.

If $a = x_0 \leq x_1 \leq x_2 \leq \dots \leq x_n = b$

, then finite ordered set $P = \{a = x_0, x_1, \dots, x_n\}$ is called a partition of $[a, b]$

$n+1$ points x_0, x_1, \dots, x_n are called partition points of P . Then n closed

sub-intervals $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$

are called segments of the partition P

Notation we use Δx_i to denote the i th interval and also its length.

Thus $\Delta x_i = x_i - x_{i-1}$

$$\sum_{i=1}^n \Delta x_i = \Delta x_1 + \Delta x_2 + \Delta x_3 + \dots + \Delta x_n = b - a$$

Norm of a Partition

The maximum of the lengths of the sub-intervals of a partition P is called the norm or mesh of partition P and is denoted by

$\|P\|$ or $\mu(P)$. OR The length of the largest of the sub-intervals of a partition P is called norm of P

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$$\begin{aligned}
 \text{Thus } \|P\| &= \max \{ \Delta x_i : i=1, 2, \dots, n \} \\
 &= \max \{ x_i - x_{i-1} : i=1, 2, \dots, n \} \\
 &= \max_{i=1}^n \Delta x_i
 \end{aligned}$$

Refinement of Partition

If P_1, P_2 are two partitions of $[a, b]$ and $P_1 \subseteq P_2$, then P_2 is called a refinement of partition P_1 .

Thus if P_2 is finer than P_1 , then every point of P_1 is used in P_2 and P_2 has at least one additional point.

Common Refinement

If P_1, P_2 are two partitions of $[a, b]$, then $P_1 \cup P_2$ is called common refinement of P_1 & P_2 because

$$P_1 \subseteq P_1 \cup P_2 \text{ \& } P_2 \subseteq P_1 \cup P_2$$

Note (1) partition is also known as dissection or net

(2) By changing the partition points the partition can be changed and hence there can be an infinite no of partitions of the interval $[a, b]$

(3) The set of all partitions of $[a, b]$ will be denoted by $P[a, b]$

Upper And Lower Darboux Sums

Let $f: [a, b] \rightarrow \mathbb{R}$ be a bounded function
i.e. f is real valued bound function defined on $[a, b]$
and $P = \{a = x_0, x_1, x_2, \dots, x_n\}$ be a partition
of $[a, b]$.

$\therefore f$ is bounded on $[a, b]$

$\therefore f$ is bounded on each sub-interval

Let $M = \sup f$ on $[a, b]$

$m = \inf f$ on $[a, b]$

$M_i = \sup f$ on $[x_{i-1}, x_i]$

$m_i = \inf f$ on $[x_{i-1}, x_i]$

Then upper and lower Darboux sums of f
corresponding to partition P are denoted by
 $U(P, f)$ or $U(f, P)$ & $L(P, f)$ or $L(f, P)$
and are defined as

$$U(P, f) = \sum_{i=1}^n M_i \Delta x_i$$

$$L(P, f) = \sum_{i=1}^n m_i \Delta x_i$$

Remarks # Clearly the sums $U(P, f)$ & $L(P, f)$
depend upon function f and partition P and
do exist for every bounded function (f)

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Oscillatory Sum & Oscillation

Let f be real valued function defined and bounded on $[a, b]$ and $P = \{x_0, x_1, x_2, \dots, x_n\}$ be a partition of $[a, b]$.

$$\text{Let } M_i = \sup f \text{ on } [x_{i-1}, x_i]$$

$$m_i = \inf f \text{ on } [x_{i-1}, x_i]$$

$$\text{Then } U(P, f) - L(P, f) = \sum_{i=1}^n M_i \Delta x_i - \sum_{i=1}^n m_i \Delta x_i$$

$$= \sum_{i=1}^n (M_i - m_i) \Delta x_i$$

$$= \sum_{i=1}^n O_i \Delta x_i$$

where $O_i = M_i - m_i$ denotes the oscillation of f in $[x_{i-1}, x_i]$ and $\sum_{i=1}^n O_i \Delta x_i$ is called oscillatory sum of f corresponding to partition P and is denoted by $\omega(P, f)$

$$\therefore O_i = M_i - m_i \geq 0$$

$$\therefore \omega(P, f) \geq 0$$

Explanation of Upper & Lower Darboux Sums

Consider a simple partition $P = \{x_0, x_1, x_2, x_3, x_4, x_5 = b\}$ and suppose that f has a simple graph as shown.

- (a) $L(P, f)$ = total area in fig (1) i.e. it is the total area of inscribed rectangles & curve $y = f(x)$ from $x = a$ to $x = b$

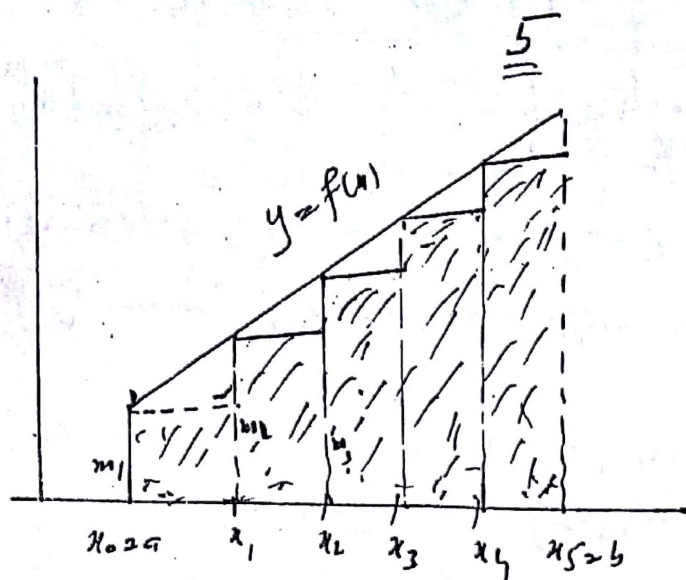
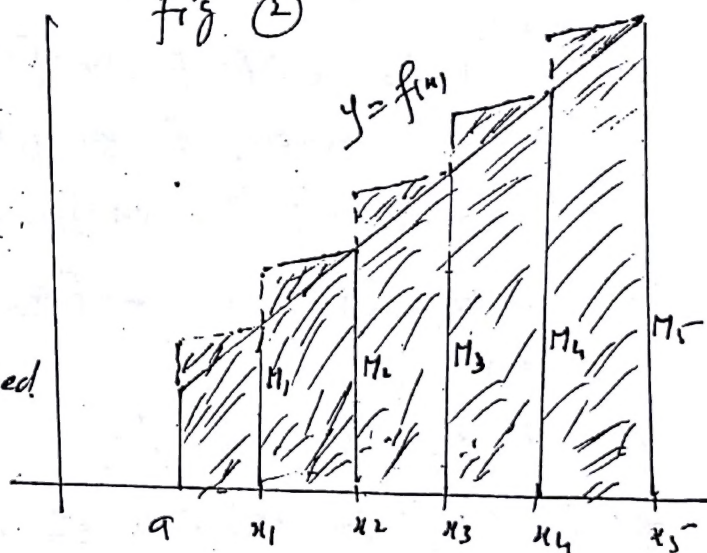


Fig ①

It approximate area under the curve from below.

(b) $U(P, f)$

\geq total shaded area in fig ② i.e. it is the total area of circumscribed rectangles of the curve $y = f(x)$ from



$x=a$ to $x=b$ and it approximate the area under f from above.

(c) If A is the exact area under the graph of f from $x=a$ to $x=b$, then

$$L(P, f) \leq A \leq U(P, f)$$

Note # When norm (mesh) of partition P is decreased by increasing the no of points of division (intermediate) points, lower sum increases and upper sum decreases. Also

$$L(P, f) \leq U(P, f)$$

Properties of Darboux Sums # 6

Theorem 1 # If $f: [a, b] \rightarrow \mathbb{R}$ is bounded & P is any partition of, then

$$m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a)$$

where m, M are infimum and supremum of f on $[a, b]$

Proof let $P = \{a = x_0, x_1, x_2, \dots, x_n = b\} \in \mathcal{C}$
a partition of $[a, b]$

$\therefore f$ is bounded on $[a, b]$

$\therefore f$ is bounded on each sub-interval $[x_{i-1}, x_i]$

Let $m_i = \inf f$ on $[x_{i-1}, x_i]$

$M_i = \sup f$ on $[x_{i-1}, x_i]$

$M = \sup f$ on $[a, b]$

$m = \inf f$ on $[a, b]$

Then

$$m \leq m_i \leq M_i \leq M$$

$$\Rightarrow m \Delta x_i \leq m_i \Delta x_i \leq M_i \Delta x_i \leq M \Delta x_i$$

$$\Rightarrow m \sum_{i=1}^n \Delta x_i \leq \sum_{i=1}^n m_i \Delta x_i \leq \sum_{i=1}^n M_i \Delta x_i \leq M \sum_{i=1}^n \Delta x_i$$

$$m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a)$$

$\Rightarrow L(P, f)$ & $U(P, f)$ are bounded if f is bounded. $\rightarrow ①$

Note # By considering all partitions of $[a, b]$ we get a set U of upper sums and set L of lower sums. The inequality ① shows that both these are bounded and each set has supremum and infimum.

2) Theorem If $f: [a, b] \rightarrow \mathbb{R}$ is a bounded function and P is any partition of $[a, b]$

$$(i) L(P, f) \leq U(P, f) \quad (ii) L(P, -f) = -U(P, f)$$

and

$$U(P, -f) = -L(P, f)$$

Note If $A \subseteq \mathbb{R}$, then let $-A = \{x \in \mathbb{R} \mid -x \in A\}$

A is bounded $\Leftrightarrow -A$ is bounded.

and $\sup(-A) = -\inf A$ & $\inf(-A) = -\sup A$

now if $f \in \mathcal{B}[a, b]$, then $-f \in \mathcal{B}[a, b]$

Proof Let $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ be any

partition of $[a, b]$. let m_i, M_i be \inf & \sup

f on $[x_{i-1}, x_i]$ & M, m be \sup & \inf of

f on $[a, b]$. Then

$$m_i \leq M_i \quad i=1, 2, 3, \dots, n$$

$$\Rightarrow m_i \Delta x_i \leq M_i \Delta x_i \quad i=1, 2, 3, \dots, n$$

$$\Rightarrow \sum m_i \Delta x_i \leq \sum_{i=1}^n M_i \Delta x_i$$

$$\Rightarrow L(P, f) \leq U(P, f)$$

$\therefore f$ is bounded on $[a, b]$

$\therefore -f$ is bounded on $[a, b]$

$$\sup(-f(x)) = -\inf f(x)$$

$$x \in [x_{i-1}, x_i]$$

$$= -m_i$$

$$\inf(-f(x)) = -\sup f(x)$$

$$x \in [x_{i-1}, x_i]$$

$$= -M_i$$

Hence

$$U(P, -f) = \sum_{i=1}^n -m_i \Delta x_i = -L(P, f)$$

$$\begin{aligned} L(P, -f) &= \sum_{i=1}^n -M_i \Delta x_i = - \sum_{i=1}^n M_i \Delta x_i \\ &= - U(P, f) \end{aligned}$$

3) Theorem # If P_1, P_2 are any two partitions of $[a, b]$ and $P_1 \subseteq P_2$, then

(i) $L(P_2, f) \geq L(P_1, f)$ lower sum cannot decrease with refinement of partition

(ii) $U(P_2, f) \leq U(P_1, f)$ The upper sum cannot increase by the refinement of partition

(iii) $L(P_1, f) \leq L(P_2, f) \leq U(P_2, f) \leq U(P_1, f)$

(iv) $U(P_1, f) - L(P_1, f) \geq U(P_2, f) - L(P_2, f)$

(v) $\omega(P_1, f) \geq \omega(P_2, f)$

Proof (i) Suppose that $P_1 = \{x_0 = a, x_1, x_2, \dots, x_n = b\}$ and P_2 be a partition with k additional points than those of P_1 such that

$$x_{i-1} < c_i < x_i \quad \forall i = 1, 2, \dots, n$$

Let $P_2' = \{c_i\} \cup P_1$ where $x_{i-1} < c_i < x_i$

let $m_i = \inf f$ on $[x_{i-1}, x_i]$

$$\omega_1 = \inf f \text{ on } [x_{i-1}, c_i]$$

$$\omega_2 = \inf f \text{ on } [c_i, x_i]$$

Then clearly

$$m_i \leq \omega_1 \quad \& \quad m_i \leq \omega_2$$

$$\begin{aligned} L(P_2', f) - L(P_1, f) &= \sum_{i=1}^n m_i \Delta x_i + \omega_1 (c_i - x_{i-1}) \\ &\quad + \omega_2 (x_i - c_i) + \sum_{i=i+1}^n m_i \Delta x_i \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i=1}^{i-1} m_i \Delta x_i + w_1 (c_1 - x_{i-1}) + w_2 (x_i - c_1) \\
 &+ \sum_{i=i+1}^n m_i \Delta x_i - \sum_{i=1}^{i-1} m_i \Delta x_i - m_i (x_i - x_{i-1}) \\
 &\quad - \sum_{i=i+1}^n m_i \Delta x_i
 \end{aligned}$$

$$= w_1 (c_1 - x_{i-1}) + w_2 (x_i - c_1) - m_i (x_i - x_{i-1})$$

$$= w_1 (c_1 - x_{i-1}) + w_2 (x_i - c_1) - m_i [x_i - c_1 + c_1 - x_{i-1}]$$

$$\Rightarrow w_1 (\omega_1 - m_i) (c_1 - x_{i-1}) + (\omega_2 - m_i) (x_i - c_1)$$

$$\therefore \omega_1 - m_i \geq 0 \quad c_1 - x_{i-1} \geq 0 \quad \omega_2 - m_i \geq 0 \quad x_i - c_1 \geq 0$$

$$\Rightarrow L(P_2', f) - L(P_1, f) \geq 0$$

$$\Rightarrow L(P_2', f) \geq L(P_1, f) \rightarrow \textcircled{1}$$

let $P_2'' = \{c_2\} \cup P'$, then by above argument

$$L(P'', f) \geq L(P_1', f) \rightarrow \textcircled{2}$$

Let $P_2''' = \{c_3\} \cup P''$

$$L(P_2''', f) \geq L(P_2'', f) \rightarrow \textcircled{3}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\text{let } P_2^k = P_2^{k-1} \cup \{c_k\} \geq P_2^{k-1}$$

$$L(P_2^k, f) \geq L(P_2^{k-1}, f) \rightarrow \textcircled{k}$$

Adding all these inequalities we get

$$L(P_2^k, f) \geq L(P_1, f)$$

$$\Rightarrow L(P_2, f) \geq L(P_1, f)$$

$$\begin{aligned}
 (ii) \quad U(P_1, f) &= \overset{L^0}{U(P_2', f)} \\
 &= \sum_{i=1}^n M_i \Delta x_i - \left[\sum_{i=1}^{i-1} M_i \Delta x_i + W_1 (c_1 - x_{i-1}) \right. \\
 &\quad \left. + W_2 (x_{i-1} - c_1) + \sum_{i=i+1}^n M_i \Delta x_i \right]
 \end{aligned}$$

$$\text{Where } W_1 = \sup f \text{ on } [x_{i-1}, c_1]$$

$$W_2 = \sup f \text{ on } [c_1, x_i]$$

$$\begin{aligned}
 &= \sum_{i=1}^{i-1} M_i \Delta x_i + M_i (x_i - x_{i-1}) + \sum_{i=i+1}^n M_i \Delta x_i \\
 &\quad - \sum_{i=1}^{i-1} M_i \Delta x_i - W_1 (c_1 - x_{i-1}) - W_2 (x_{i-1} - c_1) \\
 &\quad - \sum_{i=i+1}^n M_i \Delta x_i
 \end{aligned}$$

$$= M_i (x_i - x_{i-1}) - W_1 (c_1 - x_{i-1}) - W_2 (x_{i-1} - c_1)$$

$$= M_i [x_i - c_1 + c_1 - x_{i-1}] - W_1 (c_1 - x_{i-1}) - W_2 (x_{i-1} - c_1)$$

$$= (M_i - W_2) (x_i - c_1) + (M_i - W_1) (c_1 - x_{i-1}) \geq 0$$

$$\begin{aligned}
 \therefore M_i - W_2 &\geq 0 & M_i - W_1 &\geq 0 & x_i - c_1 &\geq 0 \\
 c_1 - x_{i-1} &\geq 0
 \end{aligned}$$

$$\Rightarrow U(P_1, f) \geq U(P_2', f) \rightarrow \textcircled{1}$$

Let $P_2'' = P_2' \cup \{c_2\}$, Then

$$U(P_2', f) \geq U(P_2'', f) \rightarrow \textcircled{2}$$

Let $P_2''' = P_2'' \cup \{c_3\}$, Then

$$U(P_2'', f) \geq U(P_2''', f) \rightarrow \textcircled{3}$$

Let $P_2^h = P_2^{h-1} \cup \{c_n\} = P_2$, Then

$$U(P_2^{h-1}, f) \geq U(P_2^h, f) \rightarrow \textcircled{4}$$

Adding all these ⁴ inequalities, we get

$$U(P_1, f) \geq U(P_2^k, f) = U(P_2, f)$$

$$\Rightarrow U(P_1, f) \geq U(P_2, f)$$

(iii) Now $U(P_1, f) \geq U(P_2, f)$

$$\boxed{\begin{array}{l} L(P_1, f) \leq L(P_2, f) \\ -L(P_1, f) \geq -L(P_2, f) \end{array}}$$

$$L(P_1, f) \leq L(P_2, f)$$

$$L(P_2, f) \leq U(P_2, f)$$

Combining these

$$L(P_1, f) \leq L(P_2, f) \leq U(P_2, f) \leq U(P_1, f)$$

(iv) $U(P_1, f) \geq U(P_2, f) \rightarrow \textcircled{1}$

$$L(P_1, f) \leq L(P_2, f)$$

$$-L(P_1, f) \geq -L(P_2, f) \rightarrow \textcircled{2}$$

Adding $\textcircled{1}$ & $\textcircled{2}$

$$U(P_1, f) - L(P_1, f) \geq U(P_2, f) - L(P_1, f)$$

OR

$$L(P_1, f) \leq L(P_2, f) \leq U(P_2, f) \leq U(P_1, f)$$

$$\Rightarrow U(P_1, f) - L(P_1, f) \geq U(P_2, f) - L(P_2, f)$$

{ i.e. on the no lineic Distance between $U(P_2, f)$ & $L(P_2, f)$ cannot exceed the distance between $U(P_1, f)$ & $L(P_1, f)$ }

(iv) It is another form of (iv)

Corollary: If P_2 is a refinement of P_1 ,
 containing p points more than P_1 and
 $|f(x)| \leq k \quad \forall x \in [a, b]$, then

$$i) \quad L(P_1, f) \leq L(P_2, f) \leq L(P_1, f) + 2pk\delta$$

$$ii) \quad U(P_1, f) \geq U(P_2, f) \geq U(P_1, f) - 2pk\delta$$

$$(iii) \quad \omega(P_1, f) - \omega(P_2, f) \leq 4pk\delta$$

where $\|P_1\| = \delta$

Proof # Let P_2 consists of just one point
 c_1 more than $P_1 = \{a, x_0, x_1, x_2, \dots, x_n = b\}$
 and $x_{n-1} < c_1 < x_n$, then

$$P_2' = \{a, x_0, x_1, x_2, \dots, x_{n-1}, c_1, x_n, \dots, x_n = b\}$$

Let m_n', m_n'' & m_n be infimum of f in
 the intervals $[x_{n-1}, c_1]$, $[c_1, x_n]$ & $[x_{n-1}, x_n]$
 respectively & M_n', M_n'', M_n be supremum of f
 on these intervals respectively

$$\because |f(x)| \leq k \quad \forall x \in [a, b]$$

$$\therefore -k \leq f(x) \leq k \quad \forall x \in [a, b]$$

$$\Rightarrow -k \leq m_n \leq m_n' \leq k$$

$$-k \leq m_n \leq m_n'' \leq k$$

$$\Rightarrow 0 \leq m_n' - m_n \leq 2k$$

$$\& \quad 0 \leq m_n'' - m_n \leq 2k$$

$$\text{Also } -k \leq M_n' \leq M_n \leq k$$

$$-k \leq M_n'' \leq M_n \leq k$$

$$\Rightarrow 0 \leq M_n - M_n'' \leq 2k$$

$$(i) \quad L(P_2', f) - L(P_1, f) = m_n'(c - x_{n-1}) + m_n''(x_n - c) - m_n(x_n - x_{n-1})$$

$$\begin{aligned}
 &= m'_k(c_1 - x_{k-1}) + m''_k(x_k - c_1) - m_k[x_k - c_1 + c_1 - x_{k-1}] \\
 &\geq (m'_k - m_k)(c_1 - x_{k-1}) + (m''_k - m_k)(x_k - c_1) \\
 &\leq 2k(c_1 - x_{k-1}) + 2k(x_k - c_1) = 2k(x_k - x_{k-1}) \\
 &\leq 2k\delta \quad \therefore x_k - x_{k-1} \leq \|P_1\| = \delta
 \end{aligned}$$

$$\Rightarrow L(P_2, f) \leq L(P_1, f) + 2k\delta \rightarrow \textcircled{1}$$

If P_2 contains p points more than P_1 , then introducing the additional points one by one and proceeding as above we have.

$$\begin{aligned}
 P_2'' &= P_2' \cup \{c_2\}, \text{ we get} \\
 L(P_2'', f) &\leq L(P_2', f) + 2k\delta \rightarrow \textcircled{2}
 \end{aligned}$$

$$\begin{aligned}
 P_2''' &= P_2'' \cup \{c_3\} \\
 L(P_2''', f) &\leq L(P_2'', f) + 2k\delta \rightarrow \textcircled{3}
 \end{aligned}$$

$$\begin{aligned}
 P_2 = P_2^p &= P_2^{p-1} \cup \{c_p\} \text{ \& } \\
 L(P_2^p, f) &\leq L(P_2^{p-1}, f) + 2k\delta \rightarrow \textcircled{p}
 \end{aligned}$$

Adding all these inequalities, we get

$$\begin{aligned}
 L(P_2^p, f) &\leq L(P_1, f) + 2kp\delta \\
 \Rightarrow L(P_2, f) &\leq L(P_1, f) + 2kp\delta
 \end{aligned}$$

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$$\begin{aligned}
 \text{(ii)} \quad & U(P_1, f) - U(P_2, f) \\
 &= M_k(x_k - x_{k-1}) - [M'_k(c_1 - x_{k-1}) + M''_k(x_k - c_1)] \\
 &= M_k[(x_k - c_1) + (c_1 - x_{k-1})] - M'_k(c_1 - x_{k-1}) \\
 &\quad - M''_k(x_k - c_1) \\
 &= (M_k - M''_k)(x_k - c_1) + (M_k - M'_k)(c_1 - x_{k-1}) \\
 &\leq 2k(x_k - c_1) + 2k(c_1 - x_{k-1})
 \end{aligned}$$

$$\leq 2k(x_2 - x_{2-1}) \stackrel{14}{\leq} 2k\delta$$

$$\leq 2k\delta \quad \text{where } x_2 - x_{2-1} \leq \|P_1\| \leq \delta$$

$$\Rightarrow U(P_2', f) \geq U(P_1, f) - 2k\delta \rightarrow (1)$$

Similarly

$$U(P_2'', f) \geq U(P_2', f) - 2k\delta \rightarrow (2)$$

$$U(P_2''', f) \geq U(P_2'', f) - 2k\delta \rightarrow (3)$$

$$\vdots$$

$$U(P_2^{p-1}, f) \geq U(P_2^{p-2}, f) - 2k\delta \rightarrow (p-1)$$

$$U(P_2^p, f) \geq U(P_2^{p-1}, f) - 2k\delta \rightarrow (p)$$

Adding all these inequalities we get

$$U(P_2^p, f) \geq U(P_1, f) - 2kp\delta$$

$$U(P_2, f) \geq U(P_1, f) - 2kp\delta$$

Also

$$U(P_1, f) \geq U(P_2, f)$$

$$\Rightarrow U(P_1, f) \geq U(P_2, f) \geq U(P_1, f) - 2kp\delta$$

$$(iii) \quad U(P_1, f) - U(P_2, f) \leq 2kp\delta$$

$$\& \quad L(P_2, f) - L(P_1, f) \leq 2kp\delta$$

Adding these we get

$$[U(P_1, f) - L(P_1, f)] - [U(P_2, f) - L(P_2, f)] \leq 4kp\delta$$

$$\Rightarrow \omega(P, f) - \omega(P_2, f) \leq 4kp\delta$$

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5) # Theorem For any two partitions P_1, P_2 of $[a, b]$

$$(i) L(P_1, f) \leq U(P_2, f)$$

$$(ii) L(P_2, f) \leq U(P_1, f)$$

i.e. no lower sum can exceed any upper sum or no upper sum can ever be less than any lower sum.

Proof # Let $P = P_1 \cup P_2$

$$\text{Then } L(P_1, f) \leq L(P, f) \text{ \& } U(P, f) \leq U(P_2, f)$$

$$\text{Also } L(P, f) \leq U(P, f)$$

Combining, we have.

$$L(P_1, f) \leq L(P, f) \leq U(P, f) \leq U(P_2, f)$$

$$\Rightarrow L(P_1, f) \leq U(P_2, f)$$

$$\text{Also } L(P_2, f) \leq L(P, f) \text{ \& } U(P, f) \leq U(P_1, f)$$

$$\text{ \& } L(P, f) \leq U(P, f)$$

Combining we have.

$$L(P_2, f) \leq L(P, f) \leq U(P, f) \leq U(P_1, f)$$

$$\Rightarrow L(P_2, f) \leq U(P_1, f)$$

6) Theorem # If $f: [a, b] \rightarrow \mathbb{R}$, $g: [a, b] \rightarrow \mathbb{R}$ are bounded functions and P is any partition of $[a, b]$, then

$$(i) U(P, f+g) \leq U(P, f) + U(P, g)$$

$$(ii) L(P, f+g) \geq L(P, f) + L(P, g)$$

$$(iii) \omega(P, f+g) \leq \omega(P, f) + \omega(P, g)$$

Proof # Let $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ be any partition of $[a, b]$

$\therefore f, g$ are bounded on $[a, b]$

$\therefore f+g$ is bounded on $[a, b]$

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$$\begin{aligned}
 \text{Let } M_k &= \sup(f+g) \text{ on } [x_{k-1}, x_k] \\
 m_k &= \inf(f+g) \text{ on } [x_{k-1}, x_k] \\
 M_k' &= \sup f \text{ on } " " \\
 m_k' &= \inf f \text{ on } " " \\
 M_k'' &= \sup g \text{ on } " " \\
 m_k'' &= \inf g \text{ on } " "
 \end{aligned}$$

$$\begin{aligned}
 \therefore M_k', M_k'' &\text{ are supremum of } f, g \text{ on } [x_{k-1}, x_k] \\
 \Rightarrow f(x) &\leq M_k' \text{ \& } g(x) \leq M_k'' \quad \forall x \in [x_{k-1}, x_k] \\
 \Rightarrow f(x) + g(x) &\leq M_k' + M_k'' \quad \forall x \in [x_{k-1}, x_k] \\
 \Rightarrow (f+g)(x) &\leq M_k' + M_k'' \quad \forall x \in [x_{k-1}, x_k] \\
 \Rightarrow M_k' + M_k'' &\text{ is an upper bound of } f+g \\
 &\text{ on } [x_{k-1}, x_k].
 \end{aligned}$$

But M_k is the least upper bound of $(f+g)$ on $[x_{k-1}, x_k]$

$$\begin{aligned}
 \Rightarrow M_k &\leq M_k' + M_k'' \quad \text{on } [x_{k-1}, x_k] \\
 &\quad \forall k \\
 \Rightarrow M_k \Delta x_k &\leq M_k' \Delta x_k + M_k'' \Delta x_k \quad \forall k.
 \end{aligned}$$

$$\Rightarrow \sum_{k=1}^n M_k \Delta x_k \leq \sum_{k=1}^n M_k' \Delta x_k + \sum_{k=1}^n M_k'' \Delta x_k$$

$$\Rightarrow U(P, f+g) \leq U(P, f) + U(P, g)$$

(ii) $\therefore m_k', m_k''$ are the ~~least~~ for the infimum of f, g on $[x_{k-1}, x_k]$

$$\begin{aligned}
 \therefore f(x) &\geq m_k' \quad \forall x \in [x_{k-1}, x_k] \quad \forall k \\
 g(x) &\geq m_k'' \quad " \quad " \quad "
 \end{aligned}$$

$$f(x) + g(x) \geq m'_k + m''_k \quad \forall x \in [x_{k-1}, x_k]$$

$$\Rightarrow (f+g)(x) \geq m'_k + m''_k \quad \text{'' '' ''}$$

$\Rightarrow m'_k + m''_k$ is a lower bound of $f+g$ on $I_k = [x_{k-1}, x_k]$.

But m_k is the greatest lower bound of $f+g$ on $I_k = [x_{k-1}, x_k]$

$$\Rightarrow m_k \geq m'_k + m''_k$$

$$\Rightarrow m_k \Delta x_k \geq m'_k \Delta x_k + m''_k \Delta x_k$$

$$\Rightarrow \sum_{k=1}^n m_k \Delta x_k \geq \sum_{k=1}^n m'_k \Delta x_k + \sum_{k=1}^n m''_k \Delta x_k$$

$$\Rightarrow L(P, f+g) \geq L(P, f) + L(P, g)$$

$$\begin{aligned} \text{(iii)} \quad \omega(P, f+g) &= U(P, f+g) - L(P, f+g) \\ &\leq [U(P, f) + U(P, g)] - [L(P, f) + L(P, g)] \\ &\leq [U(P, f) - L(P, f)] + [U(P, g) - L(P, g)] \end{aligned}$$

$$\omega(P, f+g) \leq \omega(P, f) + \omega(P, g)$$

7) Theorem # Let f, g be defined and bounded on $[a, b]$ and P be any partition of $[a, b]$

Then
(i) $U(cf, P) = c U(P, f)$ for any real constant $c \geq 0$

(ii) $L(P, cf) = c L(P, f)$ if $c < 0$

(iii) $U(P, f) \leq U(P, |f|)$

Proof (i) let $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$
be a partition of $[a, b]$, M_i, m_i be sup & inf. of f on $[x_{i-1}, x_i]$

$$\text{Then } \sup(cf) = c \sup f = c M_i \quad \text{on } [x_{i-1}, x_i]$$

$$\Rightarrow M_i' = \sup(cf) = c M_i$$

$$\begin{aligned} U(P, cf) &= \sum_{i=1}^n \cancel{c f} M_i' \Delta x_i \\ &= \sum_{i=1}^n c M_i \Delta x_i = c \sum_{i=1}^n M_i \Delta x_i \\ &= c U(P, f) \end{aligned}$$

$$(ii) \quad \text{Let } c = -c_1 \quad \text{where } c_1 > 0$$

$$\begin{aligned} M_i' = \sup(cf) &= \sup(-c_1 f) \\ &= -\inf(c_1 f) \\ &= -c_1 m_i \quad \text{where } m_i = \inf f \quad \text{on } [x_{i-1}, x_i] \\ &= c m_i' \end{aligned}$$

$$\begin{aligned} U(P, cf) &= \sum_{i=1}^n M_i' \Delta x_i \\ &= - \sum_{i=1}^n c m_i' \Delta x_i = c \sum_{i=1}^n m_i' \Delta x_i \\ &= c L(P, f) \end{aligned}$$

$$(ii) \quad \therefore f(x) \leq |f(x)| \quad \forall x \in [x_{i-1}, x_i]$$

$$\Rightarrow \sup f(x) \leq \sup |f(x)| \quad \forall x \in [x_{i-1}, x_i]$$

$$\Rightarrow M_i \leq M_i' \Rightarrow \sum M_i \Delta x_i \leq \sum M_i' \Delta x_i$$

$$\Rightarrow U(P, f) \leq U(P, |f|)$$

$$\text{Similarly } L(P, f) \leq L(P, |f|)$$

Upper & Lower Riemann Integral

Let $f: [a, b] \rightarrow \mathbb{R}$ be bounded function.
Then for every partition P of $[a, b]$

$$m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a)$$

where m, M are the infimum and supremum of f on $[a, b]$

\Rightarrow The set $\{U(P, f) : P \in P[a, b]\}$ of upper sums is bounded below by $m(b-a)$ and therefore has a greatest lower bound.

The set $\{L(P, f) : P \in P[a, b]\}$ of lower sums is bounded above by $M(b-a)$ and therefore has the least upper bound.

Lower Riemann integral is defined as $\sup \{L(P, f) : P \in P[a, b]\}$ or $\sup_P L(P, f)$

and is denoted by $\int_a^b f dx$

$$\text{Thus } \int_a^b f dx = \sup_P L(P, f)$$

Upper Riemann integral is defined as the inf $\{U(P, f) : P \in P[a, b]\}$ and is denoted by $\int_a^b f dx$.

$$\text{Thus } \int_a^b f dx = \inf_P U(P, f)$$

20 Riemann Integral

A bounded real valued function defined on $[a, b]$ is said to be Riemann integrable on $[a, b]$ if its lower and upper Riemann integrals are equal i.e. if

$$\int_a^b f dx = \int_a^b f dx.$$

The common value of these integrals is called the Riemann integral of f on $[a, b]$ and is denoted by $\int_a^b f dx$.

Note (1) Riemann integral is based on the notion of bounds and is subject to two conditions (i) f is bounded on the interval and (ii) the interval is closed & finite.

(2) The family of all bounded functions which are Riemann integrable on the closed interval $[a, b]$ is denoted by $R[a, b]$.

(3) f is R-integrable \Rightarrow (i) f is bounded on $[a, b]$ (ii) $\int_a^b f dx = \int_a^b f dx = \int_a^b f dx$

Theorem # The upper and lower integrals are always defined for every bounded function on $[a, b]$.

Proof # $\because f$ is bounded on $[a, b]$
 \therefore supremum and infimum of f on $[a, b]$ exist.

Let M, m be supremum and infimum of f on $[a, b]$. Then

$$m \leq f(x) \leq M \quad \forall x \in [a, b]$$

Let $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ be any partition of $[a, b]$ and

$$M_i = \sup f \text{ on } [x_{i-1}, x_i]$$

$$m_i = \inf f \text{ on } [x_{i-1}, x_i]$$

Then $M_i \leq M$ & $m \leq m_i \quad \forall i$

$$L(P, f) = \sum_{i=1}^n m_i \Delta x_i$$

$$\geq \sum_{i=1}^n m \Delta x_i \quad (\because m_i \geq m)$$

$$\geq m \sum_{i=1}^n \Delta x_i$$

$$\geq m(b-a) \rightarrow \textcircled{1}$$

$$U(P, f) = \sum_{i=1}^n M_i \Delta x_i$$

$$\leq \sum_{i=1}^n M \Delta x_i \quad (\because M_i \leq M)$$

$$\leq M \sum_{i=1}^n \Delta x_i$$

$$\leq M(b-a) \rightarrow \textcircled{2}$$

$$\text{Also } L(P, f) = \sum m_i \Delta x_i \leq \sum M_i \Delta x_i = U(P, f)$$

$$L(P, f) \leq U(P, f) \rightarrow \textcircled{3}$$

Combining $\textcircled{1}, \textcircled{2}$ & $\textcircled{3}$

$$m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a) \quad \forall P$$

Hence the sets $\{U(P, f) : P \in P[a, b]\}$ &

$\{L(P, f) : P \in P[a, b]\}$ are bounded sets

and hence attain inf & sup on $[a, b]$

\Rightarrow upper and lower integrals exist for a bounded function f on $[a, b]$

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Theorem # If f is real valued bounded function on $[a, b]$, then

$$\int_a^b f dx \leq \int_a^b f dx$$

Proof # Let P_1, P_2 be two partitions of $[a, b]$
 Then $L(P_1, f) \leq U(P_2, f)$

This is true for each $P_1 \in P[a, b]$.

Keeping P_2 fixed, then set $\{L(P_1, f) : P_1 \in P[a, b]\}$ has an upper bound. Also

$$\sup \{L(P_1, f) : P_1 \in P[a, b]\} = \int_a^b f dx$$

Since supremum \leq any upper sum

$$\therefore \int_a^b f dx \leq U(P_2, f) \quad \forall P_2 \in P[a, b]$$

The set $\{U(P_2, f) : P_2 \in P[a, b]\}$ has a lower bound $\int_a^b f dx$

$$\text{But } \inf \{U(P_2, f) : P_2 \in P[a, b]\} = \int_a^b f dx$$

Since any lower bound \leq infimum.

$$\therefore \int_a^b f dx \leq \int_a^b f dx$$

Theorem # If $f \in R[a, b]$, then

$$(i) \quad m(b-a) \leq \int_a^b f dx \leq M(b-a) \quad \text{if } b \geq a$$

$$(ii) \quad m(b-a) \geq \int_a^b f dx \geq M(b-a) \quad \text{if } b \leq a$$

where m, M are infimum and supremum of f on $[a, b]$

Proof # For $a = b$ the result is trivial

If $b \geq a$, then for every $P \in P[a, b]$, we have

$$m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a) \rightarrow \textcircled{1}$$

$$\text{Now } \sup_P L(P, f) \stackrel{23}{=} \int_a^b f dx = \int_a^b f dx \quad \because f \in R[a, b]$$

$$\Rightarrow L(P, f) \leq \sup_P L(P, f) = \int_a^b f dx \rightarrow (2)$$

$$\text{Also } \inf_P U(P, f) = \int_a^b f dx = \int_a^b f dx$$

$$\Rightarrow \inf_P U(P, f) \leq U(P, f)$$

$$\Rightarrow \int_a^b f dx \leq U(P, f) \rightarrow (3)$$

From ①, ② & ③

$$m(b-a) \leq L(P, f) \leq \int_a^b f dx \leq U(P, f) \leq M(b-a)$$

$$\Rightarrow m(b-a) \leq \int_a^b f dx \leq M(b-a)$$

If $b < a$, then $a > b$

\therefore interchanging in the above result

$$m(a-b) \leq \int_b^a f dx \leq M(a-b)$$

$$\Rightarrow -m(b-a) \leq -\int_a^b f dx \leq -M(b-a)$$

$$\Rightarrow m(b-a) \geq \int_a^b f dx \geq M(b-a)$$

Darboux Theorem

If f is a bounded real function on $[a, b]$, then for each $\epsilon > 0$, $\exists \delta > 0$ such that

$$(i) U(P, f) < \int_a^b f(x) dx + \epsilon$$

$$(ii) L(P, f) > \int_a^b f(x) dx - \epsilon \quad \text{for each } P \in \mathcal{P}[a, b] \text{ with } \|P\| < \delta$$

Proof (i) Let $\epsilon > 0$.

By definition $\int_a^b f dx = \sup_P U(P, f)$

$\therefore \exists$ a partition $P_1 = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ such that

$$U(P_1, f) < \int_a^b f dx + \frac{\epsilon}{2} \quad \longrightarrow \textcircled{1}$$

$\therefore f$ is bounded on $[a, b]$

let $\sup f = M$ on $[a, b]$

Suppose P_1 has k points other than a & b

and let $\delta = \frac{\epsilon}{4kM}$.

Let P be any partition such that $\|P\| < \delta$
and $P_2 = P_1 \cup P$

$$\begin{aligned} \text{Then } U(P, f) - U(P_2, f) &\leq 2kM\|P\| \\ &< 2kM\delta = 2kM \cdot \frac{\epsilon}{4kM} \\ &= \frac{\epsilon}{2} \end{aligned}$$

As P_2 is a refinement of P having
at most k more points than P

$$\Rightarrow U(P, f) < U(P_2, f) + \frac{\epsilon}{2} \leq U(P_1, f) + \frac{\epsilon}{2}$$

$$\begin{aligned} \Rightarrow U(P, f) &< U(P_1, f) + \frac{\epsilon}{2} \\ &< \int_a^b f dx + \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad \text{by } \textcircled{1} \end{aligned}$$

$$\Rightarrow U(P, f) < \int_a^b f dx + \epsilon$$

$$(ii) \text{ By definition } \int_a^b f(x) dx = \sup \left\{ L(P, f) : P \in \mathcal{P}[a, b] \right\}$$

\therefore For each $\epsilon > 0$ there exists a partition

$P_1 = \{a = x_0, x_1, x_2, \dots, x_p = b\}$ such that

$$L(P_1, f) > \int_a^b f dx - \frac{\epsilon}{2} \quad \longrightarrow \textcircled{1}$$

Suppose P_1 has k points other than a & b

and let $\delta = \frac{\epsilon}{4kM}$, where $M = \sup f$ on $[a, b]$

Let P be a partition with $\|P\| < \delta$. Then

P may contain some or none of points of P_1

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If $P_2 = P \cup P_1$, Then P_2 is finer than P and contains at most k additional points.)

$$\begin{aligned} L(P_2, f) &\leq L(P, f) + 2kM\|P_1\| \\ &\leq L(P, f) + 2kM\delta \end{aligned}$$

$$\Rightarrow L(P, f) + 2kM\delta \geq L(P_2, f) \geq L(P_1, f)$$

$$\begin{aligned} \Rightarrow L(P, f) + 2kM\delta &\geq L(P_1, f) \\ &\geq \int_a^b f dx - \epsilon/2 \end{aligned}$$

$$\Rightarrow L(P, f) \geq \int_a^b f dx - \epsilon/2 - 2kM\delta$$

$$\begin{aligned} \Rightarrow L(P, f) &\geq \int_a^b f dx - \epsilon/2 - 2kM \cdot \frac{\epsilon}{4kM} \\ &\geq \int_a^b f dx - \epsilon/2 - \epsilon/2 = \int_a^b f dx - \epsilon \end{aligned}$$

Note # The definition of infimum also leads to the fact that $U(P, f) < \int_a^b f dx + \epsilon$

but this implies that for every $\epsilon > 0$ there exist at least one partition P with this property. The importance of Darboux's theorem is the existence of an infinite no of partitions P with $\|P\| < \delta$ where δ is a +ve number depending upon ϵ

The Integral as a Limit of Sums (Riemann Sums)

The nos M_i, m_i which appear in upper and lower sums are not necessarily the values of the function f

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(There are values of f if f is continuous). We shall now show that $\int_a^b f dx$ can also be considered as the limit of a sequence of sums in which M_i, m_i are replaced by the values of f .

Corresponding to partition P of $[a, b]$, let us choose points t_1, t_2, \dots, t_n such that $x_{i-1} \leq t_i \leq x_i$ and consider the sum

$$S(P, f) = \sum_{i=1}^n f(t_i) \Delta x_i$$

The sum $S(P, f)$ is called a Riemann sum of f , over $[a, b]$ relative to P

t_i are arbitrary

we say that $S(P, f)$ converges to A as $\|P\| \rightarrow 0$

i.e.
$$\lim_{\|P\| \rightarrow 0} S(P, f) = A$$

if, for every $\epsilon > 0$, $\exists \delta > 0$ such that

$$|S(P, f) - A| < \epsilon$$

For every partition $P = \{x_0 = a, x_1, x_2, \dots, x_n = b\}$ of $[a, b]$ with norm $\|P\| < \delta$ and for every choice of points t_i in $[x_{i-1}, x_i]$

Definition # A function f is said to be integrable on $[a, b]$ if $\lim S(P, f)$ exist as $\|P\| \rightarrow 0$ and then

$$\lim_{\|P\| \rightarrow 0} S(P, f) = \int_a^b f dx$$

For any partition P and any selection of intermediate points $t_i \in [x_{i-1}, x_i]$ $i=1, 2, \dots, n$

$$L(P, f) \leq S(P, f) \leq U(P, f)$$

Riemann-Stieltjes Integral w.r.t 27

An Increasing Integrator

Upper & Lower Stieltjes (or Riemann Stieltjes

Sums

Let f be bounded function on $[a, b]$ and $\alpha(x)$ be defined as monotonically increasing on $[a, b]$. Corresponding to each partition P we write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

$\therefore \alpha$ is monotonically increasing

$$\therefore \Delta\alpha_i \geq 0$$

The upper and lower Stieltjes or Riemann Stieltjes sum of f , w.r.t α are denoted by $U(P, f, \alpha)$ and $L(P, f, \alpha)$ and are defined as

$$U(P, f, \alpha) = \sum_{i=1}^n M_i \Delta\alpha_i$$

$$L(P, f, \alpha) = \sum_{i=1}^n m_i \Delta\alpha_i$$

where

$$M_i = \sup \{ f(x) : x_{i-1} \leq x \leq x_i \}$$

$$m_i = \inf \{ f(x) : x_{i-1} \leq x \leq x_i \}$$

Note $\sum_{i=1}^n \Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$

$$= \alpha(x_1) - \alpha(x_0) + \alpha(x_2) - \alpha(x_1) + \dots + \alpha(x_n) - \alpha(x_{n-1})$$

$$= \alpha(x_n) - \alpha(x_0) = \alpha(b) - \alpha(a)$$

Properties of Stieltjes Sums

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1. Theorem # If f is bounded on $[a, b]$, α is defined and monotonically increasing on $[a, b]$, then for any partition P of $[a, b]$,

$$m[\alpha(b) - \alpha(a)] \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq M[\alpha(b) - \alpha(a)]$$

Proof # Let $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$

be a partition of $[a, b]$ and

$$M_i = \sup f(x) \quad x_{i-1} \leq x \leq x_i$$

$$m_i = \inf f(x) \quad x_{i-1} \leq x \leq x_i$$

$$M = \sup f(x) \quad x_{i-1} \leq x \leq x_i$$

$$m = \inf f(x) \quad x_{i-1} \leq x \leq x_i$$

Then

$$m \leq m_i \leq M_i \leq M$$

$$\Rightarrow m \Delta \alpha_i \leq m_i \Delta \alpha_i \leq M_i \Delta \alpha_i \leq M \Delta \alpha_i$$

$$\Rightarrow m \sum_{i=1}^n \Delta \alpha_i \leq \sum_{i=1}^n m_i \Delta \alpha_i \leq \sum_{i=1}^n M_i \Delta \alpha_i \leq M \sum_{i=1}^n \Delta \alpha_i$$

$$m[\alpha(b) - \alpha(a)] \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq M[\alpha(b) - \alpha(a)]$$

This relation holds for all partitions of $[a, b]$.

\Rightarrow The sets $\{U(P, f, \alpha) : P \in P[a, b]\}$ and

$\{L(P, f, \alpha) : P \in P[a, b]\}$ are bounded.

2) Theorem # If $f: [a, b] \rightarrow \mathbb{R}$ be bounded on $[a, b]$ and α be defined and \uparrow on $[a, b]$, then for any partition P of $[a, b]$

$$(i) L(P, f, \alpha) \leq U(P, f, \alpha) \quad (ii) L(P, -f, \alpha) = -U(P, f, \alpha)$$

Proof # Let $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$

and $m_i = \inf f(x) \quad x_{i-1} \leq x \leq x_i$

$M_i = \sup f(x) \quad x_{i-1} \leq x \leq x_i$

$M = \sup f(x) \quad a \leq x \leq b$

$m = \inf f(x) \quad a \leq x \leq b$

Then $m_i \leq M_i \quad \forall i$

$\Rightarrow m_i \Delta x_i \leq M_i \Delta x_i \quad \because \Delta x_i \geq 0$

$\Rightarrow \sum_{i=1}^n m_i \Delta x_i \leq \sum_{i=1}^n M_i \Delta x_i$

$\Rightarrow L(P, f, \alpha) \leq U(P, f, \alpha)$

$\therefore f$ is bounded on $[a, b]$

$\therefore -f$ is also bounded on $[a, b]$

$\sup[-f(x)] = -\inf f(x) \quad \forall x \in [x_{i-1}, x_i]$
 $= -m_i$

$\inf[-f(x)] = -\sup f(x) \quad \forall x \in [x_{i-1}, x_i]$
 $= -M_i$

Hence

$U(P, -f, \alpha) = \sum_{i=1}^n -m_i \Delta x_i = -L(P, f, \alpha)$

$L(P, -f, \alpha) = \sum_{i=1}^n -M_i \Delta x_i = -\sum_{i=1}^n M_i \Delta x_i$
 $= -U(P, f, \alpha)$

3 Theorem # If P_1, P_2 are any two partitions of $[a, b]$ & $P_1 \subseteq P_2$, then

(i) $L(P_2, f, \alpha) \geq L(P_1, f, \alpha)$

(ii) $U(P_2, f, \alpha) \leq U(P_1, f, \alpha)$

$$(iii) \quad L(P_1, f, \alpha) \leq L(P_2, f, \alpha) \leq U(P_2, f, \alpha) \leq U(P_1, f, \alpha)$$

$$(iv) \quad U(P_1, f, \alpha) - L(P_1, f, \alpha) \geq U(P_2, f, \alpha) - L(P_2, f, \alpha)$$

Proof # Let $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$

and P_2 has k , additional points c_1, c_2, \dots, c_k such that

$$x_{i-1} < c_i < x_i \quad \forall i = 1, 2, \dots, n$$

$$\text{Let } P_2' = P_1 \cup \{c_i\} \quad \text{where } x_{i-1} < c_i < x_i$$

$$\text{Let } m_i = \inf f \text{ on } [x_{i-1}, x_i]$$

$$\omega_1 = \inf f \text{ on } [x_{k-1}, c_1]$$

$$\omega_2 = \inf f \text{ on } [x_k, c_1, x_{k+1}]$$

Then clearly

$$m_i \leq \omega_1 \quad \& \quad m_i \leq \omega_2$$

$$L(P_2', f, \alpha) - L(P_1, f, \alpha) = \sum_{i=1}^{k-1} m_i \Delta x_i + \omega_1 [c_1 - x_{k-1}]$$

$$+ \omega_2 (x_{k+1} - c_1) + \sum_{i=k+1}^n m_i \Delta x_i$$

$$- \sum_{i=1}^n m_i \Delta x_i$$

$$= \sum_{i=1}^{k-1} m_i \Delta x_i + \omega_1 [c_1 - x_{k-1}] + \omega_2 [x_{k+1} - c_1] + \sum_{i=k+1}^n m_i \Delta x_i$$

$$- \sum_{i=1}^{k-1} m_i \Delta x_i - m_i (x_{k+1} - x_{k-1}) - \sum_{i=k+1}^n m_i \Delta x_i$$

$$= \omega_1 (c_1 - x_{k-1}) + \omega_2 (x_{k+1} - c_1) - m_i (x_{k+1} - x_{k-1})$$

$$= \omega_1 (c_1 - x_{k-1}) + \omega_2 (x_{k+1} - c_1) - m_i [(x_{k+1} - c_1) + (c_1 - x_{k-1})]$$

$$\begin{aligned}
 &= \omega_1(c_1 - x_{k-1}) + \omega_2(x_k - c_1) - m_i(x_k - c_1) - m_i(c_1 - x_{k-1}) \\
 &= (\omega_1 - m_i)(c_1 - x_{k-1}) + \omega_2(x_k - c_1) \\
 &\because \omega_1 - m_i \geq 0 \quad c_1 - x_{k-1} \geq 0 \quad \omega_2 - m_i \geq 0 \\
 &\quad x_k - c_1 \geq 0
 \end{aligned}$$

$$\Rightarrow L(P_2', f, \alpha) - L(P_1, f, \alpha) \geq 0$$

$$\Rightarrow L(P_2', f, \alpha) \geq L(P_1, f, \alpha) \longrightarrow \textcircled{1}$$

Let $P_2'' = \{c_2\} \cup P'$, then as above.

$$L(P_2'', f, \alpha) \geq L(P_1', f, \alpha) \longrightarrow \textcircled{2}$$

Let $P_2''' = \{c_3\} \cup P_2''$

$$L(P_2''', f, \alpha) \geq L(P_2'', f, \alpha) \longrightarrow \textcircled{3}$$

Let $P_2^k = P_2^{k-1} \cup \{c_k\} = P_2$

$$L(P_2, f, \alpha) \geq L(P_2^{k-1}, f, \alpha) \longrightarrow \textcircled{k}$$

Adding all these inequalities we get

$$L(P_2, f, \alpha) \geq L(P_1, f, \alpha)$$

$$(ii) U(P_1, f, \alpha) - U(P_2', f, \alpha)$$

$$\begin{aligned}
 &= \sum_{i=1}^n M_i \Delta \alpha_i - \left[\sum_{i=1}^{k-1} M_i \Delta \alpha_i + W_1(c_1 - x_{k-1}) \right. \\
 &\quad \left. + W_2(x_k - c_1) + \sum_{i=k+1}^n M_i \Delta \alpha_i \right]
 \end{aligned}$$

Where $W_1 = \sup f$ on $[x_{k-1}, c_1]$

$W_2 = \sup f$ on $[c_1, x_k]$

$$\begin{aligned}
&= \sum_{i=1}^{r-1} M_i \Delta \alpha_i + M_r (\overset{32}{x_r} - x_{r-1}) + \sum_{i=r+1}^n M_i \Delta \alpha_i \\
&\quad - \sum_{i=1}^{r-1} M_i \Delta \alpha_i - W_1 (c_1 - x_{r-1}) - W_2 (x_r - c_1) \\
&\quad - \sum_{i=r+1}^n M_i \Delta \alpha_i \\
&= M_r (x_r - x_{r-1}) - W_1 (c_1 - x_{r-1}) - W_2 (x_r - c_1) \\
&= M_r (x_r - c_1 + c_1 - x_{r-1}) - W_1 (c_1 - x_{r-1}) - W_2 (x_r - c_1) \\
&= (M_r - W_2)(x_r - c_1) + (M_r - W_1)(c_1 - x_{r-1}) \geq 0 \\
&\quad \therefore M_r - W_2 \geq 0 \quad M_r - W_1 \geq 0 \quad x_r - c_1 \geq 0 \\
&\quad \quad \quad c_1 - x_{r-1} \geq 0
\end{aligned}$$

$$\Rightarrow U(P_1, f, \alpha) \geq U(P_2', f, \alpha) \longrightarrow \textcircled{1}$$

Let $P_2'' = P_2' \cup \{c_2\}$, then

$$U(P_2', f, \alpha) \geq U(P_2'', f, \alpha) \longrightarrow \textcircled{2}$$

Let $P_2''' = P_2'' \cup \{c_3\}$, then

$$U(P_2'', f, \alpha) \geq U(P_2''', f, \alpha) \longrightarrow \textcircled{3}$$

$P_2^k = P_2^{k-1} \cup \{c_k\} = P_2$, then

$$U(P_2^{k-1}, f, \alpha) \geq U(P_2^k, f, \alpha) \longrightarrow \textcircled{k}$$

Adding all these inequalities, we get

$$U(P_1, f, \alpha) \geq U(P_2^k, f, \alpha) = U(P_2, f, \alpha)$$

$$\therefore U(P_1, f, \alpha) \geq U(P_2, f, \alpha)$$

Proof # Let $P = P_1 \cup P_2$

Then $L(P_1, f, \alpha) \leq L(P, f, \alpha)$

and $U(P_1, f, \alpha) \leq U(P, f, \alpha)$

Also $L(P, f, \alpha) \leq U(P, f, \alpha)$

and $L(P_2, f, \alpha) \leq L(P, f, \alpha)$

$U(P, f, \alpha) \leq U(P_1, f, \alpha)$

Also $L(P, f, \alpha) \leq U(P, f, \alpha)$

Combining, we have.

$$U(P_2, f, \alpha) \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq U(P_1, f, \alpha)$$

$$\Rightarrow L(P_2, f, \alpha) \leq U(P_1, f, \alpha)$$

Again.

$$L(P_1, f, \alpha) \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq U(P_2, f, \alpha)$$

$$\Rightarrow L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$$

Theorem # Let f, g be bounded functions on $[a, b]$, α be defined & \uparrow on $[a, b]$, Then for any partition P of $[a, b]$, we have

$$1) U(P, f+g) \leq U(P, f) + U(P, g)$$

$$2) L(P, f+g) \geq L(P, f, \alpha) + L(P, g, \alpha)$$

Proof # Let $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$

be any partition of $[a, b]$

f, g are bounded on $[a, b]$

(iii) Now $U(P_1, f, \alpha) \geq U(P_2, f, \alpha)$

$$L(P_1, f, \alpha) \leq L(P_2, f, \alpha)$$

$$L(P_2, f, \alpha) \leq U(P_2, f, \alpha)$$

Combining

$$L(P_1, f, \alpha) \leq L(P_2, f, \alpha) \leq U(P_2, f, \alpha) \leq U(P_1, f, \alpha)$$

(iv) $U(P_1, f, \alpha) \geq U(P_2, f, \alpha) \rightarrow \textcircled{1}$

$$L(P_1, f, \alpha) \leq L(P_2, f, \alpha)$$

$$\Rightarrow -L(P_1, f, \alpha) \geq -L(P_2, f, \alpha) \rightarrow \textcircled{2}$$

$\textcircled{1} + \textcircled{2} \Rightarrow$

$$U(P_1, f, \alpha) - L(P_1, f, \alpha) \geq U(P_2, f, \alpha) - L(P_2, f, \alpha)$$

OR

$$L(P_1, f, \alpha) \leq L(P_2, f, \alpha) \leq U(P_2, f, \alpha) \leq U(P_1, f, \alpha)$$

$$\Rightarrow U(P_1, f, \alpha) - L(P_1, f, \alpha) \geq U(P_2, f, \alpha) - L(P_2, f, \alpha)$$

[i.e. on the number line distance b/w $U(P_2, f, \alpha)$ & $L(P_2, f, \alpha)$ can not exceed the distance bet $U(P_1, f, \alpha)$ & $L(P_1, f, \alpha)$]

4) # Theorem # For any two partitions P of $[a, b]$

(i) $L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$

(ii) $L(P_2, f, \alpha) \leq U(P_1, f, \alpha)$

no upper sum can ever be less than any lower sum.

$\therefore f+g$ is bounded

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$$\begin{aligned} \text{Let } M_i &= \sup(f+g) && \text{on } [x_{i-1}, x_i] \\ m_i &= \inf(f+g) && \text{on } [x_{i-1}, x_i] \\ M_i' &= \sup f && \text{on } " " \\ m_i' &= \inf f && \text{on } " " \\ M_i'' &= \sup g && \text{on } " " \\ m_i'' &= \inf g && \text{on } " " \end{aligned}$$

$\therefore M_i', M_i''$ are supremum of f, g on $[x_{i-1}, x_i]$

$$\Rightarrow f(x) \leq M_i' \quad \forall x \in [x_{i-1}, x_i]$$

$$g(x) \leq M_i'' \quad \forall x \in [x_{i-1}, x_i]$$

$$\Rightarrow f(x) + g(x) \leq M_i' + M_i'' \quad \forall x \in [x_{i-1}, x_i]$$

$$\Rightarrow (f+g)(x) \leq M_i' + M_i'' \quad \forall x \in [x_{i-1}, x_i]$$

$\Rightarrow M_i' + M_i''$ is an upper bound of $(f+g)$

But M_i is the least upper bound of $(f+g)$ on $[x_{i-1}, x_i]$

$$\Rightarrow M_i \leq M_i' + M_i''$$

$$\Rightarrow M_i \Delta x_i \leq M_i' \Delta x_i + M_i'' \Delta x_i$$

$$\Rightarrow \sum_{i=1}^n M_i \Delta x_i \leq \sum_{i=1}^n M_i' \Delta x_i + \sum_{i=1}^n M_i'' \Delta x_i$$

$$\Rightarrow U(P, f+g, \alpha) \leq U(P, f, \alpha) + U(P, g, \alpha)$$

(ii) $\therefore m_i', m_i''$ are the minimum of f, g

on $[x_{i-1}, x_i]$

$$\therefore f(x) \geq m_i'$$

$$g(x) \geq m_i''$$

$$\forall x \in [x_{i-1}, x_i]$$

$$" " "$$

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$$36 \quad f(x) + g(x) \geq m'_i + m''_i \quad \forall x \in [x_{i-1}, x_i]$$

$$\Rightarrow (f+g)(x) \geq m'_i + m''_i \quad \text{'' ''}$$

$$\Rightarrow m'_i + m''_i \text{ is a lower bound of } f+g \text{ on } [x_{i-1}, x_i]$$

But m_i the greatest lower bound of $f+g$ on $[x_{i-1}, x_i]$

$$\Rightarrow m_i \geq m'_i + m''_i$$

$$\Rightarrow m_i \Delta x_i \geq m'_i \Delta x_i + m''_i \Delta x_i$$

$$\Rightarrow \sum_{i=1}^n m_i \Delta x_i \geq \sum_{i=1}^n m'_i \Delta x_i + \sum_{i=1}^n m''_i \Delta x_i$$

$$L(P, f+g) \geq L(P, f) + L(P, g)$$

6: Theorem Let f be bounded on $[a, b]$, c be defined and \uparrow on $[a, b]$. Then for any partition P of $[a, b]$

$$i) \quad U(P, cf, \alpha) = c U(P, f, \alpha) \quad \text{for any real } c \geq 0$$

$$ii) \quad U(P, cf, \alpha) = c L(P, f, \alpha) \quad \text{if } c < 0$$

$$iii) \quad |U(P, f, \alpha)| \leq U(P, |f|, \alpha)$$

Proof # (i) Let $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ be a partition of $[a, b]$, M_i, m_i be \sup & \inf of f on $[x_{i-1}, x_i]$

$$\text{Then } \sup(cf) = c \sup f = c M_i \quad \text{on } [x_{i-1}, x_i]$$

$$M'_i = c M_i$$

$$U(P, cf, \alpha) = \sum_{i=1}^n (c M_i) \Delta x_i$$

$$= c \sum_{i=1}^n M_i \Delta x_i = c U(P, f, \alpha)$$

(ii) Let $c = -c_1$ where $c_1 > 0$

$$\begin{aligned} M_i' &= \sup(cf) = \sup(-c_1 f) \\ &= -\inf(c_1 f) \\ &= -c_1 \inf f \\ &= c m_i \quad \text{on } [x_{i-1}, x_i] \end{aligned}$$

$$\begin{aligned} U(P, cf, \alpha) &= \sum M_i' \Delta \alpha_i \\ &= \sum c m_i \Delta \alpha_i \\ &= c L(P, f, \alpha) \end{aligned}$$

(iii) Note $|f|(u) = |f(u)|$

$$\therefore |f| = \max(f, -f)$$

$$\therefore f(u) \leq |f(u)| = |f|(u) \quad \forall u \in [x_{i-1}, x_i]$$

$$-f(u) \leq |f(u)| = |f|(u) \quad \forall u \in [x_{i-1}, x_i]$$

Also $-|f(u)| \leq f(u) \leq |f(u)| = |f|(u)$

$$\Rightarrow \text{Let } M_i' = \sup |f| \text{ over } [x_{i-1}, x_i]$$

$$-M_i' \leq \sup\{f(u) : u \in [x_{i-1}, x_i]\}$$

$$\leq \sup\{|f(u)| : u \in [x_{i-1}, x_i]\} = M_i'$$

$$\Rightarrow -M_i' \leq M_i \leq M_i'$$

$$\Rightarrow -\sum M_i' \Delta \alpha_i \leq \sum_{i=1}^n M_i \Delta \alpha_i \leq \sum_{i=1}^n M_i' \Delta \alpha_i$$

$$\Rightarrow -U(P, |f|, \alpha) \leq U(P, f, \alpha) \leq U(P, |f|, \alpha)$$

$$\Rightarrow |U(P, f, \alpha)| \leq U(P, |f|, \alpha) \text{ proved!}$$

Riemann Stieltjes³⁸ Integral

Let f real valued bounded on $[a, b]$
and α be defined and \uparrow on $[a, b]$.

Let $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ be
a partition of $[a, b]$. we write

$$\Delta x_i = \alpha(x_i) - \alpha(x_{i-1})$$

$$\text{If } M_i = \sup f \text{ on } [x_{i-1}, x_i]$$

$$m_i = \inf f \text{ on } [x_{i-1}, x_i]$$

Then upper and lower Riemann Stieltjes
Sums are defined by

$$U(P, f, \alpha) = \sum_{i=1}^n M_i \Delta x_i$$

$$L(P, f, \alpha) = \sum_{i=1}^n m_i \Delta x_i$$

If m, M are infimum and supremum
of f on $[a, b]$, we have

$$m \leq m_i \leq M_i \leq M$$

$$\Rightarrow m \sum \Delta x_i \leq \sum m_i \Delta x_i \leq \sum M_i \Delta x_i \leq M \sum \Delta x_i$$

$$m [\alpha(b) - \alpha(a)] \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq M [\alpha(b) - \alpha(a)]$$

This inequality shows that both sets of lower
and upper sets are bound and each set
sup & inf.

The upper Riemann Stieltjes Integral
w.r. α on $[a, b]$ is defined as

$$\int_a^b f d\alpha = \sup \{ U(P, f, \alpha) : P \in P[a, b] \}$$

$$= \sup_P U(P, f, \alpha)$$

and lower Riemann Stieltjes integral is defined as

$$\int_a^b f d\alpha = \inf \{ L(P, f, \alpha) : P \in P[a, b] \}$$

$$= \inf_P L(P, f, \alpha)$$

where the inf & sup are taken over all possible partitions of $[a, b]$

$$\text{If } \int_a^b f d\alpha = \int_a^b f d\alpha$$

, then f is Riemann Stieltjes integrable and symbolically we write

$$f \in R_\alpha[a, b] \text{ or } f \in (R, [a, b]) \text{ or } R(\alpha)$$

$$\text{over } [a, b] \text{ or } f \in RS_\alpha \text{ on } [a, b] \text{ or } f \in RS[\alpha; a, b]$$

Their common value is denoted by

$$\int_a^b f d\alpha \text{ or } \int_a^b f^{(n)} d\alpha^{(n)}$$

Note #. Riemann Stieltjes integral is a general case of Riemann integral because if $\alpha'(x) = x$, then $\Delta\alpha_i = \Delta x_i$ and the Stieltjes integral reduces to Riemann integral

In other words Riemann integral is a special case of Stieltjes integral. Since $\int_a^b f(x) d\alpha(x)$, depends upon f, α, a, b but not on x we write $\int_a^b f d\alpha$

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Letter α is a "dummy variable" 400

Example #10 If $\alpha(u) = k \quad \forall u \in [a, b]$ &
 f is any bounded function on $[a, b]$, Then
 $f \in R(\alpha)$

(b) # If α is an increasing function
on $[a, b]$ and $f(u) = k$, then prove that
 $f \in R(\alpha)$

Sol # Let P be any partition of $[a, b]$
Then $\alpha(u) = k \quad \forall u \in [x_{i-1}, x_i]$

$$\begin{aligned} \text{and } L(P, f, \alpha) &= \sum_{i=1}^n m_i \Delta \alpha_i \\ &= \sum_{i=1}^n m_i [\alpha(x_i) - \alpha(x_{i-1})] \\ &= \sum_{i=1}^n m_i [k - k] = 0 \quad \forall P \end{aligned}$$

\Rightarrow Every lower sum is zero

Hence $\int_a^b f d\alpha = \sup_P L(P, f, \alpha) = 0$

Again

$$U(P, f, \alpha) = \sum_{i=1}^n M_i \Delta \alpha_i = 0 \quad \forall P$$

$$\Rightarrow \int_a^b f d\alpha = 0$$

Thus $\int_a^b f d\alpha = \int_a^b f d\alpha$

$$\Rightarrow f \in R(\alpha) \text{ on } [a, b]$$

(b) $\because f(u) = k \quad \forall u \in [a, b]$

$$\begin{aligned} \Rightarrow m_i &= \inf \{ f(u) : x_{i-1} \leq u \leq x_i \} \\ &= k \quad \forall i \end{aligned}$$

and $M_i = k \quad \forall p$

$$\text{Hence } U(P, f, \alpha) = \sum_{i=1}^n M_i \Delta \alpha_i$$

$$= k [\alpha(b) - \alpha(a)] \quad \forall P$$

$$\int_a^b f d\alpha = \inf_P U(P, f, \alpha)$$

Similarly $\int_a^b f d\alpha = k [\alpha(b) - \alpha(a)]$

$$\text{Thus } \int_a^b f d\alpha = \int_a^b f d\alpha$$

$$\Rightarrow f \in R(\alpha)$$

Theorem # The upper and lower Riemann-Stieltjes integrals are always defined for bounded function f

Proof # Let f be bounded function on $[a, b]$ and α be defined and \uparrow on $[a, b]$. Let M and m be supremum and infimum of f in $[a, b]$

$$\text{Then } M_i \leq M \quad \forall i$$

$$m \leq m_i \quad \forall i$$

$$\Rightarrow m \leq m_i \leq M_i \leq M$$

$$\Rightarrow \sum m_i \Delta \alpha_i \leq \sum m_i \Delta \alpha_i \leq \sum M_i \Delta \alpha_i \leq \sum M \Delta \alpha_i$$

$$\Rightarrow m [\alpha(b) - \alpha(a)] \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq M [\alpha(b) - \alpha(a)]$$

This is true for all partition

\Rightarrow The upper and lower sums form bounded sets and attain their inf & sup

\Rightarrow upper and lower integrals are always defined for bound function.

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Theorem # Let f be a bounded function on $[a, b]$ and α be defined and \uparrow on $[a, b]$, then

$$\int_a^b f d\alpha \leq \int_a^b f d\alpha.$$

Proof # Let P_1, P_2 be any two partitions of $[a, b]$. Then

$$L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$$

Keeping P_2 fixed, the set $\{L(P_1, f, \alpha) : P_1 \in P[a, b]\}$ has an upper bound Also

$$\sup_{P_1 \in P[a, b]} L(P_1, f, \alpha) = \int_a^b f d\alpha$$

\therefore Supremum \leq any upper bound

$$\int_a^b f d\alpha \leq U(P_2, f, \alpha) \quad \forall P_2 \in P[a, b]$$

\Rightarrow The set $\{U(P_2, f, \alpha) : P_2 \in P[a, b]\}$ has a lower bound $\int_a^b f d\alpha$

$$\text{But } \inf_{P_2 \in P[a, b]} U(P_2, f, \alpha) = \int_a^b f d\alpha$$

Since any lower bound \leq Infimum

$$\int_a^b f d\alpha \leq \int_a^b f d\alpha.$$

Cauchy's Criterion for Integrability #

OR

Riemann Condition #

Theorem # Let f be a bound function on $[a, b]$ and α be defined, \uparrow on $[a, b]$, then $f \in R(\alpha)$ iff for every $\epsilon > 0$, \exists a partition P of $[a, b]$

Such that

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$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

i.e. Difference b/w upper and Lower sums
Corresponding to any partition remain arbitrarily
Small

Proof# Necessary#

Let $f \in R(\alpha)$ over $[a, b]$

$$\text{Then } \int_a^b f dx = \int_a^b f dx = \int_a^b f dx$$

Let $\epsilon > 0$ be given.

$$\therefore \int_a^b f dx = \inf_P U(P, f, \alpha)$$

\therefore By definition of infimum \exists a partition P_1

Such that

$$U(P_1, f) < \int_a^b f dx + \frac{\epsilon}{2} \rightarrow (1)$$

$$\therefore \int_a^b f dx = \sup_P L(P, f, \alpha)$$

\therefore By definition of supremum \exists a partition P_2

$$L(P_2, f, \alpha) > \int_a^b f dx - \frac{\epsilon}{2} \rightarrow (2)$$

$$\therefore \int_a^b f dx = \int_a^b f dx = \int_a^b f dx$$

$$\Rightarrow U(P_1, f, \alpha) < \int_a^b f dx + \frac{\epsilon}{2} \rightarrow (3)$$

$$L(P_2, f, \alpha) > \int_a^b f dx - \frac{\epsilon}{2} \rightarrow (4)$$

Let $P = P_1 \cup P_2$. Then

$$U(P, f, \alpha) \leq U(P_1, f, \alpha) < \int_a^b f dx + \frac{\epsilon}{2}$$

$$U(P, f, \alpha) < \int_a^b f dx + \frac{44}{2} \epsilon \rightarrow (5)$$

Also

$$L(P, f, \alpha) \geq L(P_2, f, \alpha) > \int_a^b f dx - \frac{\epsilon}{2}$$

$$\Rightarrow L(P, f, \alpha) > \int_a^b f dx - \frac{\epsilon}{2}$$

$$\Rightarrow \int_a^b f dx - \frac{\epsilon}{2} < L(P, f, \alpha) \rightarrow (6)$$

Adding (5) & (6)

$$U(P, f, \alpha) + \int_a^b f dx - \frac{\epsilon}{2} < L(P, f, \alpha) + \int_a^b f dx + \frac{\epsilon}{2}$$

$$\Rightarrow U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

Converse (Sufficiency Condition)

Conversely suppose that for every $\epsilon > 0$ \exists a partition P such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

But

$$L(P, f, \alpha) \leq \int_a^b f dx \rightarrow (7)$$

$$\& \int_a^b f dx \leq U(P, f, \alpha) \rightarrow (8)$$

Adding (7) & (8)

$$L(P, f, \alpha) + \int_a^b f dx \leq U(P, f, \alpha) + \int_a^b f dx$$

$$\Rightarrow \int_a^b f dx - \int_a^b f dx \leq U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

$$\text{But } \int_a^b f dx \leq \int_a^b f dx \Rightarrow 0 \leq \int_a^b f dx - \int_a^b f dx$$

$$\Rightarrow 0 \leq \int_a^b f dx - \int_a^b f dx < \epsilon \quad \forall \epsilon > 0$$

But a non-negative no. can be less than every +ve no. if it is zero $\therefore \int_a^b f dx = \int_a^b f dx \Rightarrow f \in R(\alpha)$

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Riemann Stieltjes Integral as Limit of
Riemann Stieltjes Sum

Riemann Stieltjes Sum #

Let f be bounded function on $[a, b]$, α be defined, \uparrow on $[a, b]$. Let $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ be a partition of $[a, b]$.

Let $T = \{t_1, t_2, \dots, t_n\}$, where $t_i \in [x_{i-1}, x_i]$ for $i = 1, 2, \dots, n$. A sum of form

$$S(P, f, \alpha) = \sum_{i=1}^n f(t_i) \Delta \alpha_i$$

is called Riemann Stieltjes sum w.r.t α for the partition. It is also denoted by $S(P, T, f)$

clearly.

$$L(P, f, \alpha) \leq S(P, f, \alpha) \leq U(P, f, \alpha)$$

For any choice of points T (or t_i)

We say that

$$\lim_{\|P\| \rightarrow 0} S(P, f, \alpha) = A$$

if for every $\epsilon > 0$, $\exists \delta > 0$ such that for any partition P of $[a, b]$ with $\|P\| < \delta$ and every choice of points t_i in $[x_{i-1}, x_i]$

$$|S(P, f, \alpha) - A| < \epsilon$$

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Theorem # If $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$
holds for some P and some ϵ , Then

(a) # It holds (with same ϵ) for every refinement
of P

(b) # If $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$ holds for
some partition $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$
and if s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$
Then

$$\sum_{i=1}^n |f(s_i) - f(t_i)| \Delta x_i < \epsilon$$

Proof # (a) # Let P^* be a refinement of P

Then $L(P, f, \alpha) \leq L(P^*, f, \alpha) \rightarrow \textcircled{1}$

$$U(P^*, f, \alpha) \leq U(P, f, \alpha) \rightarrow \textcircled{2}$$

Adding $\textcircled{1}$ & $\textcircled{2}$

$$L(P, f, \alpha) + U(P^*, f, \alpha) \leq L(P^*, f, \alpha) + U(P, f, \alpha)$$

$$U(P^*, f, \alpha) - L(P^*, f, \alpha) \leq U(P, f, \alpha) - L(P, f, \alpha)$$

$$\Rightarrow U(P^*, f, \alpha) - L(P^*, f, \alpha) < \epsilon$$

This result holds for any refinement of P

(b) # Let $\sup f(x) = M_i \quad \forall x \in [x_{i-1}, x_i]$
 $\inf f(x) = m_i$

$\because s_i, t_i$ are arbitrary points in $[x_{i-1}, x_i]$
therefore both $f(s_i), f(t_i)$ lie in $[m_i, M_i]$

$$m_i \leq f(s_i) \leq M_i \rightarrow \textcircled{1}$$

$$m_i \leq f(t_i) \leq M_i \rightarrow \textcircled{2}$$

$$\Rightarrow |f(s_i) - f(t_i)| \leq M_i - m_i \quad \forall i$$

$$\begin{aligned} \Rightarrow \sum_{i=1}^n |f(s_i) - f(t_i)| \Delta x_i &\leq \sum_{i=1}^n (M_i - m_i) \Delta x_i \\ &= U(P, f, \alpha) - L(P, f, \alpha) < \epsilon \end{aligned}$$

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Theorem # If f is bounded function on $[a, b]$ and α be defined, \uparrow on $[a, b]$, then $f \in R(\alpha)$ iff for every real $\epsilon > 0$ $\exists \delta > 0$ (depending on ϵ) such that

$$\left| \sum_{i=1}^n f(t_i) \Delta \alpha_i - \int_a^b f d\alpha \right| < \epsilon$$

for every partition P with $\|P\| < \delta$ and for every choice $t_i \in [x_{i-1}, x_i]$

Proof # Let $f \in R(\alpha)$ and $\epsilon > 0$. Then Riemann condition holds and \exists a partition P of $[a, b]$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon \rightarrow (1)$$

Let P^* be a refinement of P . Then

$$U(P^*, f, \alpha) - L(P^*, f, \alpha) \leq U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

and $\|P^*\| < \delta = \|P\|$

$$\Rightarrow U(P^*, f, \alpha) - L(P^*, f, \alpha) < \epsilon \rightarrow (2)$$

Also for the same partition P , we have.

$$L(P, f, \alpha) \leq \int_a^b f d\alpha \leq U(P, f, \alpha) \rightarrow (3)$$

and for the same partition P^* , we have.

$$L(P^*, f, \alpha) \leq \sum_{i=1}^n f(t_i) \Delta \alpha_i \leq U(P^*, f, \alpha) \rightarrow (4)$$

By (3) & (4)

$$\left| \sum_{i=1}^n f(t_i) \Delta \alpha_i - \int_a^b f d\alpha \right| \leq U(P^*, f, \alpha) - L(P^*, f, \alpha) < \epsilon$$

Hence the condition is proved

Converse # Suppose ⁴⁸ that for any $\epsilon > 0 \exists \delta$

such that

$$\left| \sum_{i=1}^n f(t_i) \Delta x_i - \int_a^b f dx \right| < \epsilon$$

 for every partition $P = \{x_0, x_1, \dots, x_n\}$ with $\|P\| < \delta$ and for every choice of points $t_i \in [x_{i-1}, x_i]$
 we prove that $f \in R(\alpha)$ Let $\alpha(b) > \alpha(a)$

\therefore The condition holds

\therefore For all choices t_i, t'_i in $[x_{i-1}, x_i]$
 and for all partitions P with $\|P\| < \delta$, we

have.

$$\left| \sum_{i=1}^n f(t_i) \Delta x_i - \int_a^b f dx \right| < \frac{\epsilon}{4} \longrightarrow (5)$$

$$\left| \sum_{i=1}^n f(t'_i) \Delta x_i - \int_a^b f dx \right| < \frac{\epsilon}{4} \longrightarrow (6)$$

Since

$$M_i - m_i = \sup \{ f(x) - f(x') : x, x' \in [x_{i-1}, x_i] \}$$

Therefore by definition of supremum, we have

$$M_i - m_i - \frac{\epsilon}{2[\alpha(b) - \alpha(a)]} < f(t_i) - f(t'_i)$$

$$M_i - m_i < f(t_i) - f(t'_i) + \frac{\epsilon}{2[\alpha(b) - \alpha(a)]} \longrightarrow (7)$$

$$U(P, f, \alpha) - L(P, f, \alpha)$$

$$\begin{aligned} &= \sum_{i=1}^n (M_i - m_i) \Delta x_i < \sum_{i=1}^n \left[f(t_i) - f(t'_i) + \frac{\epsilon}{2[\alpha(b) - \alpha(a)]} \right] \Delta x_i \\ &= \sum_{i=1}^n [f(t_i) - f(t'_i)] \Delta x_i + \frac{\epsilon}{2[\alpha(b) - \alpha(a)]} \sum_{i=1}^n \Delta x_i \\ &= \sum f(t_i) \Delta x_i - \sum f(t'_i) \Delta x_i + \frac{\epsilon}{2} \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^n f(t_i) \Delta x_i - \int_a^b f dx - \sum_{i=1}^n f(t'_i) \Delta x_i + \int_a^b f dx + \epsilon/2 \\
&\leq \left| \sum_{i=1}^n f(t_i) \Delta x_i - \int_a^b f dx - \left(\sum_{i=1}^n f(t'_i) \Delta x_i - \int_a^b f dx \right) \right| + \epsilon/2 \\
&\leq \left| \sum_{i=1}^n f(t_i) \Delta x_i - \int_a^b f dx \right| + \left| \sum_{i=1}^n f(t'_i) \Delta x_i - \int_a^b f dx \right| + \epsilon/2
\end{aligned}$$

$$\begin{aligned}
&< \epsilon/4 + \epsilon/4 + \epsilon/2 = \epsilon \\
&\Rightarrow \text{Riemann condition is true} \\
&\Rightarrow f \in R(\alpha)
\end{aligned}$$

OR

Let the condition holds. We assume $\alpha(b) > \alpha(a)$.
 Let $\epsilon > 0$, \exists a $\delta > 0$ such that

$$\left| \sum_{i=1}^n f(t_i) \Delta x_i - \int_a^b f dx \right| < \epsilon/4 \quad \rightarrow (8)$$

$\forall P$, with $\|P\| < \delta$ and for every choice of points $t_i \in [x_{i-1}, x_i]$

For each i , $1 \leq i \leq n$, $\exists t'_i \in [x_{i-1}, x_i]$

such that

$$M_i - \frac{\epsilon}{4[\alpha(b) - \alpha(a)]} < f(t'_i)$$

$$\Rightarrow M_i < \frac{\epsilon}{4[\alpha(b) - \alpha(a)]} + f(t'_i)$$

$$\begin{aligned}
\Rightarrow \sum_{i=1}^n M_i \Delta x_i &< \frac{\epsilon}{4[\alpha(b) - \alpha(a)]} \sum_{i=1}^n \Delta x_i + \sum_{i=1}^n f(t'_i) \Delta x_i \\
&= \epsilon/4 + \sum_{i=1}^n f(t'_i) \Delta x_i
\end{aligned}$$

$$\Rightarrow U(P, f, \alpha) < \epsilon/4 + \sum_{i=1}^n f(t'_i) \Delta x_i \rightarrow (9)$$

Similarly for each i , $1 \leq i \leq n$ $\exists t''_i \in [x_{i-1}, x_i]$
 such that

$$f(t_i'') < m_i + \frac{\epsilon}{4[\alpha(b) - \alpha(a)]} \quad 50 \underline{49}$$

$$\Rightarrow \sum_{i=1}^n f(t_i'') \Delta \alpha_i < \sum m_i \Delta \alpha_i + \frac{\epsilon}{4}$$

$$S(P, f, \alpha) < L(P, f, \alpha) + \frac{\epsilon}{4} \rightarrow (10)$$

$$\Rightarrow -L(P, f, \alpha) < -\sum f(t_i'') \Delta \alpha_i + \frac{\epsilon}{4} \rightarrow (11)$$

Adding (9) & (11)

$$U(P, f, \alpha) - L(P, f, \alpha) < \sum_{i=1}^n f(t_i') \Delta \alpha_i + \frac{\epsilon}{2} - \sum_{i=1}^n f(t_i'') \Delta \alpha_i + \frac{\epsilon}{2}$$

$$= \sum_{i=1}^n f(t_i') \Delta \alpha_i - \int_a^b f d\alpha + \frac{\epsilon}{2} + \int_a^b f d\alpha - \sum_{i=1}^n f(t_i'') \Delta \alpha_i$$

$$\leq \left| \sum_{i=1}^n f(t_i') \Delta \alpha_i - \int_a^b f d\alpha \right| + \frac{\epsilon}{2} + \left| \sum_{i=1}^n f(t_i'') \Delta \alpha_i - \int_a^b f d\alpha \right|$$

$$< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon \quad \text{by (8)}$$

$$\Rightarrow f \in R(\alpha)$$

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Properties of Riemann-Stieltjes Integral

1# Theorem (Linearity Properties)

(a) If $f \in R(\alpha)$ on $[a, b]$, then $cf \in R(\alpha)$ for every constant c and

$$\int_a^b cf \, d\alpha = c \int_a^b f \, d\alpha.$$

(b) If $f_1 \in R(\alpha)$, $f_2 \in R(\alpha)$, then $f_1 + f_2 \in R(\alpha)$ on $[a, b]$ and

$$\int_a^b (f_1 + f_2) \, d\alpha = \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha.$$

(c) If $f_1, f_2 \in R(\alpha)$ on $[a, b]$, then $c_1 f_1 + c_2 f_2 \in R(\alpha)$ on $[a, b]$ and

$$\int_a^b (c_1 f_1 + c_2 f_2) \, d\alpha = c_1 \int_a^b f_1 \, d\alpha + c_2 \int_a^b f_2 \, d\alpha$$

Proof (a) If $c = 0$, then Theorem is obvious.

so let $c \neq 0$

$\therefore f \in R(\alpha)$

\therefore Riemann condition is True

Case I Let $c > 0$

$\therefore f \in R(\alpha)$

$\therefore \forall \epsilon > 0 \exists$ a partition P such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon/c \rightarrow \textcircled{1}$$

$$U(P, cf, \alpha) = \sum_{i=1}^n c M_i \Delta \alpha_i = c U(P, f, \alpha)$$

$$L(P, cf, \alpha) = \sum_{i=1}^n c m_i \Delta \alpha_i = c L(P, f, \alpha)$$

$$U(P, cf, \alpha) - L(P, cf, \alpha) = c [U(P, f, \alpha) - L(P, f, \alpha)]$$

$$< c \cdot \frac{\epsilon}{c} = \epsilon$$

$$\Rightarrow cf \in R(\alpha) \text{ on } [a, b]$$

$$\text{Also } \int_a^b cf \, d\alpha = \int_a^b (cf) \, d\alpha$$

$$U(P, cf, \alpha) = c U(P, f, \alpha)$$

$$\inf_P U(P, cf, \alpha) = c \inf_P U(P, f, \alpha)$$

$$\int_a^b (cf) \, d\alpha = c \int_a^b f \, d\alpha$$

$$\int_a^b (cf) \, d\alpha = c \int_a^b f \, d\alpha \quad \therefore cf \in R(\alpha) \text{ if } f \in R(\alpha)$$

Case II If $c < 0$

$$\therefore f \in R(\alpha)$$

$$\therefore \text{for } \epsilon > 0 \quad \exists \text{ a partition } P \text{ of } [a, b]$$

such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \frac{\epsilon}{|c|}$$

$$\sup (cf) = c \inf f = c m_i \quad \text{on } [x_{i-1}, x_i]$$

$$M_i' = c m_i$$

$$\inf (cf) = c \sup f = c M_i$$

$$m_i' = c M_i$$

$$U(P, cf, \alpha) = \sum_{i=1}^n M_i' \Delta \alpha_i = c \sum_{i=1}^n m_i \Delta \alpha_i$$

$$= c L(P, f, \alpha)$$

$$L(P, cf, \alpha) = \sum_{i=1}^n m_i' \Delta \alpha_i = c \sum_{i=1}^n M_i \Delta \alpha_i$$

$$= c U(P, f, \alpha)$$

$$\begin{aligned}
 U(P, cf, \alpha) - L(P, cf, \alpha) &= c L(P, f, \alpha) - c U(P, f, \alpha) \\
 &= -c [U(P, f, \alpha) - L(P, f, \alpha)] \quad 53 \\
 &= |c| [U(P, f, \alpha) - L(P, f, \alpha)] \\
 &< |c| \cdot \frac{\epsilon}{|c|} = \epsilon \quad \because |c| = -c \quad c < 0
 \end{aligned}$$

$$\Rightarrow cf \in R(\alpha)$$

$$\inf_P U(P, cf, \alpha) = \inf_P [c L(P, f, \alpha)]$$

$$= c \sup_P L(P, f, \alpha)$$

$$\int_a^b (cf) d\alpha = c \int_a^b f d\alpha$$

$$\Rightarrow \int_a^b (cf) d\alpha = c \int_a^b f d\alpha \quad \because f \in R(\alpha) \\ cf \in R(\alpha)$$

$$(b) \text{ Proof } \because f_1, f_2 \in R(\alpha)$$

\therefore for a given $\epsilon > 0 \exists$ partitions P_1, P_2

such that

$$U(P_1, f_1, \alpha) - L(P_1, f_1, \alpha) < \frac{\epsilon}{2}$$

$$U(P_2, f_2, \alpha) - L(P_2, f_2, \alpha) < \frac{\epsilon}{2}$$

Let $P = P_1 \cup P_2$. Then

$$U(P, f_1, \alpha) - L(P, f_1, \alpha) < \frac{\epsilon}{2} \rightarrow (1)$$

$$U(P, f_2, \alpha) - L(P, f_2, \alpha) < \frac{\epsilon}{2} \rightarrow (2)$$

$$\text{Let } P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$$

$$\because f_1, f_2 \in R(\alpha) \quad \therefore f_1, f_2 \text{ are bounded on } [a, b]$$

$$\Rightarrow f_1 + f_2 \text{ is bounded on } [a, b]$$

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$$\text{Let } f = f_1 + f_2$$

$$M_i = \sup f \text{ on } [x_{i-1}, x_i]$$

$$m_i = \inf f \text{ on } [x_{i-1}, x_i]$$

$$M_i' = \sup f_1 \text{ " " "}$$

$$m_i' = \inf f_1 \text{ " " "}$$

$$M_i'' = \sup f_2 \text{ " " "}$$

$$m_i'' = \inf f_2 \text{ " " "}$$

Then $f_1(u) \leq M_i' \quad \forall u \in [x_{i-1}, x_i]$

$$f_2(u) \leq M_i'' \text{ " " "}$$

$$\Rightarrow f_1(u) + f_2(u) \leq M_i' + M_i'' \text{ " " "}$$

$$\Rightarrow (f_1 + f_2)(u) \leq M_i' + M_i'' \text{ " " "}$$

$$\Rightarrow M_i' + M_i'' \text{ is an upper bound of } f_1 + f_2 \text{ on } [x_{i-1}, x_i]$$

But M_i is least upper bound of $f_1 + f_2$ on $[x_{i-1}, x_i]$

$$M_i \leq M_i' + M_i'' \quad i = 1, 2, \dots, n$$

$$\Rightarrow \sum_{i=1}^n M_i \Delta x_i \leq \sum_{i=1}^n M_i' \Delta x_i + \sum_{i=1}^n M_i'' \Delta x_i$$

$$U(P, f_1 + f_2, \alpha) \leq U(P, f_1, \alpha) + U(P, f_2, \alpha) \quad \rightarrow (3)$$

Similarly

$$L(P, f_1, \alpha) + L(P, f_2, \alpha) \leq L(P, f_1 + f_2, \alpha) \quad \rightarrow (4)$$

$$\begin{aligned} & U(P, f_1 + f_2, \alpha) + L(P, f_1, \alpha) + L(P, f_2, \alpha) \\ & \leq U(P, f_1, \alpha) + U(P, f_2, \alpha) + L(P, f_1 + f_2, \alpha) \end{aligned}$$

$$\begin{aligned}
 & U(P, f_1 + f_2, \alpha) - L(P, f_1 + f_2, \alpha) \\
 & \leq U(P, f_1, \alpha) - L(P, f_1, \alpha) + U(P, f_2, \alpha) - L(P, f_2, \alpha) \\
 & < \epsilon/2 + \epsilon/2 = \epsilon
 \end{aligned}$$

$\Rightarrow f = f_1 + f_2 \in R(\alpha)$ on $[a, b]$

$$\underline{\int_a^b (f_1 + f_2) d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha.}$$

Now let $\epsilon > 0 \exists$ partitions P_1', P_2' of $[a, b]$ such that

$$\int_a^b f_1 d\alpha - \epsilon/2 < L(P_1', f_1, \alpha)$$

$$\Rightarrow \int_a^b f_1 d\alpha - \epsilon/2 < L(P_1', f_1, \alpha) \rightarrow (5)$$

and $\int_a^b f_2 d\alpha - \epsilon/2 < L(P_2', f_2, \alpha)$

$$\int_a^b f_2 d\alpha - \epsilon/2 < L(P_2', f_2, \alpha) \rightarrow (6)$$

If $P^* = P_1' \cup P_2'$, then

$$L(P_1', f_1, \alpha) \leq L(P^*, f_1, \alpha)$$

$$L(P_2', f_2, \alpha) \leq L(P^*, f_2, \alpha)$$

By (5) & (6)

$$\int_a^b f_1 d\alpha - \epsilon/2 < L(P^*, f_1, \alpha) \rightarrow (7)$$

$$\int_a^b f_2 d\alpha - \epsilon/2 < L(P^*, f_2, \alpha) \rightarrow (8)$$

Adding (7) & (8)

$$\begin{aligned}
 \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha - \epsilon & < L(P^*, f_1, \alpha) + L(P^*, f_2, \alpha) \\
 & \leq L(P^*, f_1 + f_2, \alpha) \leq \int_a^b (f_1 + f_2) d\alpha
 \end{aligned}$$

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$\therefore c$ is an arbitrary ≈ 56

$$\therefore \int_a^b f_1 dx + \int_a^b f_2 dx \leq \int_a^b (f_1 + f_2) dx \rightarrow (9)$$

Replacing f_1 by $-f_1$, f_2 by $-f_2$ we have from (9)

$$\int_a^b -f_1(x) dx + \int_a^b -f_2(x) dx \leq \int_a^b -(f_1 + f_2) dx$$

$$\Rightarrow \int_a^b f_1 dx + \int_a^b f_2 dx \geq \int_a^b (f_1 + f_2) dx \rightarrow (10)$$

By (9) & (10)

$$\int_a^b f_1 dx + \int_a^b f_2 dx = \int_a^b (f_1 + f_2) dx$$

Cor If $f_1, f_2 \in R(\alpha)$ on $[a, b]$, $f_1 - f_2 \in R(\alpha)$

and $\int_a^b (f_1 - f_2) dx = \int_a^b f_1 dx - \int_a^b f_2 dx$

Proof # $\because f_2 \in R(\alpha)$

$$\therefore -f_2 = (-1) \cdot f_2 \in R(\alpha)$$

$$\Rightarrow f_1 + (-1)f_2 \in R(\alpha)$$

$$\Rightarrow f_1 - f_2 \in R(\alpha)$$

Also $\int_a^b (f_1 + (-1)f_2) dx = \int_a^b f_1 dx + \int_a^b -f_2 dx$

$$= \int_a^b f_1 dx - \int_a^b f_2 dx$$

(C) Proof # $\because f_1 \in R(\alpha)$ $c_1 \in \mathbb{R}$

$$\therefore c_1 f_1 \in R(\alpha) \quad \& \quad \int_a^b c_1 f_1 dx = c_1 \int_a^b f_1 dx$$

Similarly $c_2 f_2 \in R(\alpha)$

and $\int_a^b c_2 f_2 dx = c_2 \int_a^b f_2 dx$

Now $c_1 f_1 \in R(\alpha)$, $c_2 f_2 \in R(\alpha)$

$$\Rightarrow c_1 f_1 + c_2 f_2 \in R(\alpha)$$

$$\text{and } \int_a^b (c_1 f_1 + c_2 f_2) d\alpha = \int_a^b c_1 f_1 d\alpha + \int_a^b c_2 f_2 d\alpha \quad 57$$

$$= c_1 \int_a^b f_1 d\alpha + c_2 \int_a^b f_2 d\alpha$$

OR

If $c_1 = 0, c_2 = 0$, then theorem is obvious.

Let $c_1 \neq 0, c_2 \neq 0$

Case I If $c_1 > 0, c_2 > 0$

$\therefore f_1 \in R(\alpha), f_2 \in R(\alpha)$

\therefore For $\epsilon > 0$ \exists partitions P_1, P_2 s. that

$$U(P_1, f_1, \alpha) - L(P_1, f_1, \alpha) < \frac{\epsilon}{2c_1}$$

$$U(P_2, f_2, \alpha) - L(P_2, f_2, \alpha) < \frac{\epsilon}{2c_2}$$

Let $P = P_1 \cup P_2$

$$\text{Then } U(P, f_1, \alpha) - L(P, f_1, \alpha) < \frac{\epsilon}{2c_1}$$

$$U(P, f_2, \alpha) - L(P, f_2, \alpha) < \frac{\epsilon}{2c_2}$$

Let $f = c_1 f_1 + c_2 f_2$

$\text{Sup } f = \text{Sup } (c_1 f_1 + c_2 f_2) \quad \text{on } [x_{i-1}, x_i]$

Let $M_i = \text{Sup } f \quad m_i = \text{Inf } f \quad \text{on } [x_{i-1}, x_i]$

$M_i' = \text{Sup } f_1 \quad m_i' = \text{Inf } f_1 \quad \text{on } [x_{i-1}, x_i]$

$M_i'' = \text{Sup } f_2 \quad m_i'' = \text{Inf } f_2 \quad \text{on } [x_{i-1}, x_i]$

$$f_1(x) \leq M_i' \quad \forall x \in [x_{i-1}, x_i]$$

$$f_2(x) \leq M_i'' \quad \forall x \in [x_{i-1}, x_i]$$

$$\Rightarrow c_1 f_1(x) \leq c_1 M_i'$$

$$c_2 f_2(x) \leq c_2 M_i''$$

$$\Rightarrow c_1 f_1(x) + c_2 f_2(x) \leq c_1 M_i' + c_2 M_i''$$

$$\Rightarrow M_i \leq c_1 M_i' + c_2 M_i''$$

$$\Rightarrow U(P, f, \alpha) \leq c_1 U(P, f_1, \alpha) + c_2 U(P, f_2, \alpha) \rightarrow \dots$$

Similarly

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$$L(P, f, \alpha) \geq L(P, c_1 f_1, \alpha) + L(P, c_2 f_2, \alpha)$$

$$L(P, c_1 f_1, \alpha) + L(P, c_2 f_2, \alpha) \leq L(P, f, \alpha) \rightarrow (11)$$

Adding (10) & (11)

$$U(P, f, \alpha) - L(P, f, \alpha) = c_1 [U(P, f_1, \alpha) - L(P, f_1, \alpha)] + c_2 [U(P, f_2, \alpha) - L(P, f_2, \alpha)]$$

$$< c_1 \frac{\epsilon}{2c_1} + c_2 \cdot \frac{\epsilon}{2c_2} = \epsilon$$

$$\Rightarrow f = c_1 f_1 + c_2 f_2 \in R(\alpha) \text{ on } [a, b]$$

Let $\epsilon > 0$ \exists partitions P_1', P_2'' s. that

$$\int_a^b c_1 f_1 dx - \epsilon/2 < L(P_1', c f_1, \alpha) \rightarrow (12)$$

$$\int_a^b c_1 f_1 dx - \epsilon/2 < L(P_1', c f_1, \alpha) \rightarrow (12)$$

$$\int_a^b c_2 f_2 dx - \epsilon/2 < L(P_2'', c f_2, \alpha) \rightarrow (13)$$

Let $P^* = P_1' \cup P_2''$, then

$$\int_a^b c_1 f_1 dx - \epsilon/2 < L(P_1', c f_1, \alpha) < L(P^*, c f_1, \alpha)$$

$$\int_a^b c_2 f_2 dx - \epsilon/2 < L(P_2'', c f_2, \alpha) < L(P^*, c f_2, \alpha) \rightarrow (15)$$

Adding

$$\int_a^b c_1 f_1 dx + \int_a^b c_2 f_2 dx - \epsilon < L(P^*, c f_1, \alpha) + L(P^*, c f_2, \alpha) \leq L(P^*, f, \alpha) \leq \int_a^b f dx$$

$\therefore \epsilon$ is an arbitrary no

$$\int_a^b c_1 f_1 dx + \int_a^b c_2 f_2 dx \leq \int_a^b f dx$$

changing c_1, c_2, f we

$$\int_a^b c_1 f_1 dx + \int_a^b c_2 f_2 dx \geq \int_a^b f dx$$

$$\Rightarrow \int_a^b c_1 f_1 d\alpha + \int_a^b c_2 f_2 d\alpha = \int_a^b (c_1 f_1 + c_2 f_2) d\alpha$$

$$\Rightarrow c_1 \int_a^b f_1 d\alpha + c_2 \int_a^b f_2 d\alpha = \int_a^b (c_1 f_1 + c_2 f_2) d\alpha$$

Case II If $c_1 < 0, c_2 < 0$

$$\therefore f_1, f_2 \in R(\alpha)$$

\therefore for $\epsilon > 0 \quad \exists p_1', p_2'$ such that

$$U(p_1', f_1, \alpha) - L(p_1', f_1, \alpha) < \frac{\epsilon}{2|c_1|}$$

$$U(p_2', f_2, \alpha) - L(p_2', f_2, \alpha) < \frac{\epsilon}{2|c_2|}$$

$$\text{Let } P^* = p_1' \cup p_2'$$

$$\Rightarrow U(P^*, f_1, \alpha) - L(P^*, f_1, \alpha) < \frac{\epsilon}{2|c_1|}$$

$$U(P^*, f_2, \alpha) - L(P^*, f_2, \alpha) < \frac{\epsilon}{2|c_2|}$$

$$\text{Let } f = c_1 f_1 + c_2 f_2$$

$$\text{Sup } f = M_i \text{ on } [x_{i-1}, x_i], \quad \text{Inf } f = m_i$$

$$\text{Sup } c_1 f_1 = c_1 \text{Inf } f_1 = c_1 m_i$$

$$\text{Inf } c_1 f_1 = c_1 \text{Sup } f_1 = c_1 M_i$$

$$\text{Sup } c_2 f_2 = c_2 \text{Inf } f_2 = c_2 m_i$$

$$\text{Inf } c_2 f_2 = c_2 \text{Sup } f_2 = c_2 M_i$$

$$\text{Now } c_1 f_1(u) \leq c_1 m_i' \quad \forall u \in [x_{i-1}, x_i]$$

$$c_2 f_2(u) \leq c_2 m_i''$$

$$\Rightarrow c_1 f_1(u) + c_2 f_2(u) \leq c_1 m_i' + c_2 m_i''$$

$$\Rightarrow f(u) \leq c_1 m_i' + c_2 m_i''$$

$$\Rightarrow M_i \leq c_1 m_i' + c_2 m_i''$$

$$\Rightarrow U(P^*, f, \alpha) \leq c_1 L(P^*, f_1, \alpha) + c_2 L(P^*, f_2, \alpha) \rightarrow (1)$$

Similarly

$$L(P^*, f, \alpha) \geq c_1 U(P^*, f_1, \alpha) + c_2 U(P^*, f_2, \alpha) \rightarrow (2)$$

$$60 \quad c_1 U(P, f_1, \alpha) + c_2 U(P, f_2, \alpha) < L(P, f, \alpha) \rightarrow (16)$$

Adding (16) & (17)

$$U(P, f, \alpha) + c_1 U(P, f_1, \alpha) + c_2 U(P, f_2, \alpha) \\ < L(P, f, \alpha) + c_1 L(P, f_1, \alpha) + c_2 L(P, f_2, \alpha)$$

$$U(P, f, \alpha) - L(P, f, \alpha) < -c_1 [U(P, f_1, \alpha) - L(P, f_1, \alpha)] \\ - c_2 [U(P, f_2, \alpha) - L(P, f_2, \alpha)]$$

$$= |c_1| [U(P, f_1, \alpha) - L(P, f_1, \alpha)] \\ + |c_2| [U(P, f_2, \alpha) - L(P, f_2, \alpha)]$$

$$\leq |c_1| \frac{\epsilon}{2|c_1|} + |c_2| \frac{\epsilon}{2|c_2|} = \epsilon$$

$$\Rightarrow f \in R(\alpha)$$

Now it can be proved as above.

$$\int_a^b (c_1 f_1 + c_2 f_2) d\alpha = c_1 \int_a^b f_1 d\alpha + c_2 \int_a^b f_2 d\alpha$$

Case III If one of c_1 & c_2 is -ve say $c_1 < 0, c_2 > 0$

$$\therefore f_1, f_2 \in R(\alpha)$$

\therefore for $\epsilon > 0$ \exists partitions P_1, P_2 such that

$$U(P_1, f_1, \alpha) - L(P_1, f_1, \alpha) < \frac{\epsilon}{2|c_1|}$$

$$U(P_2, f_2, \alpha) - L(P_2, f_2, \alpha) < \frac{\epsilon}{2c_2}$$

$$\text{Let } P = P_1 \cup P_2$$

$$\text{Then } U(P, f_1, \alpha) - L(P, f_1, \alpha) < \frac{\epsilon}{2|c_1|}$$

$$U(P, f_2, \alpha) - L(P, f_2, \alpha) < \frac{\epsilon}{2c_2}$$

$$\sup c_1 f_1 = c_1 \inf f_1 = c_1 m_1'$$

$$\inf c_1 f_1 = c_1 \sup f_1 = c_1 M_1'$$

$$\sup c_2 f_2 = c_2 M_2''$$

$$\inf c_2 f_2 = c_2 m_2''$$

$$c_1 f_1(x) \leq c_1 m_i' \quad \forall x \in [x_{i-1}, x_i]$$

$$c_2 f_2(x) \leq c_2 M_i'' \quad \forall x \in [x_{i-1}, x_i]$$

$$\Rightarrow c_1 f_1(x) + c_2 f_2(x) \leq c_1 m_i' + c_2 M_i''$$

$$\Rightarrow M_i \leq c_1 m_i' + c_2 M_i''$$

$$U(P, f, \alpha) \leq c_1 L(P, f_1, \alpha) + c_2 U(P, f_2, \alpha)$$

$$c_1 M_i' \leq c_1 f_1(x) \quad \forall x \in [x_{i-1}, x_i]$$

$$c_2 m_i'' \leq c_2 f_2(x) \quad \forall x \in [x_{i-1}, x_i]$$

$$\Rightarrow c_1 M_i' + c_2 m_i'' \leq c_1 f_1(x) + c_2 f_2(x)$$

$$c_1 M_i' + c_2 m_i'' \leq m_i$$

$$\Rightarrow c_1 U(P, f_1, \alpha) + c_2 L(P, f_2, \alpha) \leq L(P, f, \alpha)$$

Adding

$$U(P, f, \alpha) - L(P, f, \alpha) = -c_1 [U(P, f_1, \alpha) - L(P, f_1, \alpha)] \\ + c_2 [U(P, f_2, \alpha) - L(P, f_2, \alpha)]$$

$$= |c_1| [U(P, f_1, \alpha) - L(P, f_1, \alpha)]$$

$$+ c_2 [U(P, f_2, \alpha) - L(P, f_2, \alpha)]$$

$$< |c_1| \cdot \frac{\epsilon}{2|c_1|} + c_2 \cdot \frac{\epsilon}{2c_2} = \epsilon$$

$$\Rightarrow f = c_1 f_1 + c_2 f_2 \in R(\alpha)$$

2.1) # Order Preserving Properties

Under the assumption that α is monotone increasing, several useful order preserving properties of the integral can be proved

We prove the following properties

(a) If $f_1 \in R(\alpha)$, $f_2 \in R(\alpha)$ & $f_1(x) \leq f_2(x)$ on $[a, b]$, then 62

$$\int_a^b f_1 d\alpha \leq \int_a^b f_2 d\alpha$$

(b) If $f \in R(\alpha)$, $f(x) \geq 0 \quad \forall x \in [a, b]$, then

$$\int_a^b f(x) d\alpha \geq 0$$

(c) If $f \in R(\alpha)$, then $|f| \in R(\alpha)$ and

$$\left| \int_a^b f d\alpha \right| \leq \int_a^b |f| d\alpha$$

(d) If $f \in R(\alpha)$ on $[a, b]$ & $|f(x)| \leq M$ on $[a, b]$, then

$$\left| \int_a^b f d\alpha \right| \leq M[\alpha(b) - \alpha(a)]$$

Proof # (a) Let $f(x) \geq 0 \quad \forall x \in [a, b]$
Then $M_1 \geq 0$

$$\Rightarrow U(P, f, \alpha) \geq 0$$

$$\Rightarrow \int_a^b f d\alpha \geq 0$$

$$\therefore f_1 \leq f_2 \therefore f_2 - f_1 \geq 0$$

$$\Rightarrow \int_a^b (f_2 - f_1) d\alpha \geq 0$$

$$\Rightarrow \int_a^b f_2 d\alpha - \int_a^b f_1 d\alpha \geq 0$$

$$\Rightarrow \int_a^b f_1 d\alpha \leq \int_a^b f_2 d\alpha$$

OR

$$\therefore f_1, f_2 \in R(\alpha)$$

$$\therefore \int_a^b f_1 d\alpha = \int_a^b f_1 d\alpha = \int_a^b f_1 d\alpha$$

$$\text{and } \int_a^b f_2 dx = \int_a^b f_2 dx = \int_a^b f_2 dx. \quad b3$$

$$\text{Let } \inf f_1 = m_i' \quad \text{on } [x_{i-1}, x_i]$$

$$\inf f_2 = m_i'' \quad \text{on } [x_{i-1}, x_i]$$

$$\therefore f_1(x) \leq f_2(x) \quad \forall x \in [x_{i-1}, x_i]$$

$$\Rightarrow m_i' \leq m_i''$$

$$\Rightarrow \sum m_i' \Delta x_i \leq \sum m_i'' \Delta x_i$$

$$L(P, f_1, \alpha) \leq L(P, f_2, \alpha) \quad \forall P$$

$$\Rightarrow \int_a^b f_1 dx \leq \int_a^b f_2 dx$$

$$\Rightarrow \int_a^b f_1 dx \leq \int_a^b f_2 dx$$

$$(c) \# \therefore f \in R(a), f \text{ is bounded on } [a, b]$$

$$\therefore \exists \text{ a +ve no } k, \text{ such that}$$

$$|f(x)| \leq k \Rightarrow |f|(x) \leq k \quad \forall x \in [a, b]$$

$$\Rightarrow |f| \text{ is bounded}$$

$$\text{Let } P \text{ be any partition of } [a, b]$$

$$M_i = \sup f \quad \text{on } [x_{i-1}, x_i]$$

$$m_i = \inf f \quad \text{on } [x_{i-1}, x_i]$$

$$M_i' = \sup |f| \quad \text{on } [x_{i-1}, x_i]$$

$$m_i' = \inf |f| \quad \text{on } [x_{i-1}, x_i]$$

$$||f(x)| - |f(y)|| \leq |f(x) - f(y)| \quad \forall x, y \in [x_{i-1}, x_i]$$

$$\Rightarrow ||f(x)| - |f(y)|| \leq M_i - m_i \rightarrow \textcircled{1}$$

$$\therefore M_i', m_i' \text{ denote supremum and infimum on } [x_{i-1}, x_i]$$

$$\therefore f(u) \leq M_i' \text{ \& } |f(x)| \geq m_i' \quad \forall x \in [x_{i-1}, x_i]$$

$\Rightarrow \exists \epsilon > 0$ such that

$$|f(u)| > M_i' - \epsilon \rightarrow (2)$$

$$\& |f(y)| < m_i' + \epsilon \rightarrow (3)$$

from (2) \& (3)

$$2\epsilon + |f(u)| - |f(y)| > M_i' - m_i'$$

$$M_i' - m_i' < 2\epsilon + |f(u)| - |f(y)|$$

$\therefore \epsilon$ is arbitrary

$$\therefore M_i' - m_i' \leq |f(u)| - |f(y)| \rightarrow (4)$$

Interchang $x \& y$

$$M_i' - m_i' \leq -(|f(u)| - |f(y)|) \rightarrow (5)$$

from (4) \& (5)

$$M_i' - m_i' \leq ||f(u)| - |f(y)|| \rightarrow (6)$$

from (1) \& (6)

$$M_i' - m_i' \leq M_i - m_i$$

$$\Rightarrow \sum_{i=1}^n (M_i' - m_i') \Delta x_i \leq \sum_{i=1}^n (M_i - m_i) \Delta x_i$$

$$\Rightarrow U(P, |f|, \alpha) - L(P, |f|, \alpha) \leq U(P, f, \alpha) - L(P, f, \alpha)$$

$$\Rightarrow |f| \in R(\alpha) \quad < \epsilon$$

$$\text{Since } |f| = \max\{f, -f\}$$

\therefore

$$f(x) \leq |f(x)| = |f|(x) \quad \forall x \in [a, b]$$

$$-f(x) \leq |f(x)| = |f|(x) \quad \forall x \in [a, b]$$

$$\Rightarrow \int_a^b f(x) dx \leq \int_a^b |f|(x) dx. \quad \rightarrow (7)$$

$$\& \int_a^b -f(x) dx \leq \int_a^b |f|(x) dx.$$

$$-\int_a^b f(x) dx \leq \int_a^b |f|(x) dx.$$

$$\Rightarrow \left[\int_a^b |f|(x) dx \geq \int_a^b f(x) dx \right]$$

$$\int_a^b f(x) dx \geq -\int_a^b |f|(x) dx. \quad \rightarrow (8)$$

By (7) & (8) we get

$$\left| \int_a^b f dx \right| \leq \int_a^b |f|(x) dx.$$

Remarks # The Converse of the above result is not true i.e. if $|f|$ is integrable, then f may or may not be integrable e.g. if we take

$$f(x) = \begin{cases} 1 & \text{when } x \text{ is rational} \\ -1 & \text{when } x \text{ is irrational} \end{cases}$$

$$\text{and } g(x) = x$$

$$\text{Then } \int_a^b f dx = b-a$$

$$\int_a^b f dx = (b-a)$$

So that f is not integrable.

But since $|f(x)| = 1 \quad \forall x$

$$\int_a^b |f(x)| dx = b-a \quad \text{i.e. } |f| \text{ integrable}$$

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(d) # We know that

$$\int_a^b f d\alpha \leq U(P, f, \alpha)$$

$$= \sum_i M_i \Delta \alpha_i \leq M \sum_i \Delta \alpha_i$$

$$\left| \int_a^b f d\alpha \right| \leq \int_a^b |f(u)| d\alpha$$

$$\leq M \int_a^b d\alpha = M[\alpha(b) - \alpha(a)]$$

$$\Rightarrow \left| \int_a^b f d\alpha \right| \leq M[\alpha(b) - \alpha(a)]$$

3) # Theorem # if $f \in R(\alpha)$, then $f^2 \in R(\alpha)$ on $[a, b]$

Proof # $\because f \in R(\alpha) \therefore |f| \in R(\alpha)$

$\therefore f$ is bounded on $[a, b]$

$\therefore |f|$ is bounded on $[a, b]$

Also $f^2 = |f|^2$ is bounded on $[a, b]$

If M_i, m_i sup and inf of f on $[x_{i-1}, x_i]$, then M_i^2 & m_i^2 are sup and inf of f^2 on $[x_{i-1}, x_i]$

$\therefore f$ is bounded on $[a, b]$

$\therefore \exists$ no M such that

$$|f(u)| \leq M \text{ \& } M_i \leq M, m_i \leq M$$

Let $\epsilon > 0$

$\therefore f \in R(\alpha)$ on $[a, b]$

$\therefore \exists$ a partition P of $[a, b]$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \frac{\epsilon}{2M} \rightarrow \textcircled{1}$$

$$U(P_1, f, \alpha) - L(P_1, f, \alpha) < \frac{\epsilon}{2k} \rightarrow (2)$$

$$U(P_2, f, \alpha) - L(P_2, f, \alpha) < \frac{\epsilon}{2k} \rightarrow (3)$$

If $P = P_1 \cup P_2$, then

$$U(P, f, \alpha) - L(P, f, \alpha) < \frac{\epsilon}{2k} \rightarrow (4)$$

$$U(P, g, \alpha) - L(P, g, \alpha) < \frac{\epsilon}{2k} \rightarrow (5)$$

now for the same partition P condition (1) holds & we have.

$$U(P, fg, \alpha) - L(P, fg, \alpha)$$

$$= \sum_{i=1}^n (M_i - m_i) \Delta \alpha_i$$

$$\leq \sum_{i=1}^n [k(M_i' - m_i') + k(M_i'' - m_i'')] \Delta \alpha_i$$

$$= k[U(P, f, \alpha) - L(P, f, \alpha)] + k[U(P, g, \alpha) - L(P, g, \alpha)]$$

$$< k \cdot \frac{\epsilon}{2k} + k \left(\frac{\epsilon}{2k} \right) = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$$\Rightarrow U(P, fg, \alpha) - L(P, fg, \alpha) < \epsilon$$

$$\Rightarrow fg \in R(\alpha) \text{ on } [a, b]$$

$$\therefore f \in R(\alpha), \text{ OR } g \in R(\alpha)$$

$$\Rightarrow f+g \in R(\alpha), f-g \in R(\alpha)$$

$$\Rightarrow (f+g)^2 \in R(\alpha), (f-g)^2 \in R(\alpha) \quad \therefore f \in R(\alpha) \Rightarrow f^2 \in R(\alpha)$$

$$\Rightarrow (f+g)^2 - (f-g)^2 \in R(\alpha)$$

$$\Rightarrow \frac{1}{4} [(f+g)^2 - (f-g)^2] \in R(\alpha) \Rightarrow fg \in R(\alpha)$$

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$$(c) \quad \therefore |f(x)| \geq t \quad \forall x \in [a, b]$$

$$\Rightarrow \frac{1}{|f(x)|} \leq \frac{1}{t} \quad \forall x \in [a, b]$$

$$\Rightarrow \left| \frac{1}{f(x)} \right| \leq \frac{1}{t} \quad " "$$

$$\Rightarrow \left| \frac{1}{f}(x) \right| \leq \frac{1}{t} \quad " "$$

$\Rightarrow \frac{1}{f}$ is bound

$$\text{Let } M_i' = \sup\left(\frac{1}{f}\right) \text{ on } [x_{i-1}, x_i]$$

$$m_i' = \inf\left(\frac{1}{f}\right) \quad " \quad " "$$

$$M_i = \sup\left(\frac{1}{f}\right) \quad " \quad " "$$

$$m_i = \inf\left(\frac{1}{f}\right) \quad " \quad " "$$

$$\forall x, y \in [x_{i-1}, x_i]$$

$$\left| \left(\frac{1}{f}\right)(y) - \left(\frac{1}{f}\right)(x) \right| = \frac{|f(y) - f(x)|}{|f(x)| |f(y)|}$$

$$\leq \frac{1}{t^2} |f(y) - f(x)|$$

$$\Rightarrow M_i' - m_i' \leq \frac{1}{t^2} (M_i - m_i) \quad \forall \text{ partitions} \quad \longrightarrow \textcircled{1}$$

$$\therefore f \in R(a)$$

$$\therefore \text{for } \epsilon > 0 \quad \exists \text{ a partition } p \text{ of } [a, b]$$

such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon t^2$$

$$\sum (M_i - m_i) \Delta x_i < \epsilon t^2 \quad \longrightarrow \textcircled{2}$$

Now for the same partition p , we have

$$U(P, f^2, \alpha) - L(P, f^2, \alpha) \stackrel{0}{=} b9$$

$$= \sum M_i^2 \Delta \alpha_i - \sum m_i^2 \Delta \alpha_i$$

$$= \sum_{i=1}^n (M_i^2 - m_i^2) \Delta \alpha_i$$

$$= \sum_{i=1}^n (M_i + m_i)(M_i - m_i) \Delta \alpha_i$$

$$\leq 2M \sum_{i=1}^n (M_i - m_i) \Delta \alpha_i$$

$$< 2M \cdot \frac{\epsilon}{2M} = \epsilon$$

$$\Rightarrow f^2 \in R(\alpha) \text{ on } [a, b]$$

Algebraic Properties of Integral.

4) # Theorem # Suppose that integrator is
is monotone increasing on $[a, b]$, $f, g \in R_\alpha[a, b]$

, Then

(a) $f^2 \in R_\alpha[a, b]$

(b) $fg \in R_\alpha[a, b]$

(c) If there is a no $t > 0$ such that
 $|f(x)| \geq t > 0$ i.e if f is bounded away
from zero, then $\frac{1}{f} \in R(\alpha)$,

Proof (a) Already proved.

(b) $\because f, g \in R(\alpha)$

$\therefore f, g$ are bounded on $[a, b]$

$\Rightarrow \exists$ numbers k_1, k_2 such that

$$|f(x)| \leq k_1 \quad \forall x \in [a, b]$$

$$|g(x)| \leq k_2 \quad \forall x \in [a, b]$$

Let $k = \max \{k_1, k_2\}$

Then $|f(x)| \leq k \quad \forall x \in [a, b]$
 $|g(x)| \leq k \quad \forall x \in [a, b]$

$$\Rightarrow |(fg)(x)| = |f(x)g(x)|$$

$$= |f(x)| |g(x)| \leq k^2 \quad \forall x \in [a, b]$$

Let $M_i = \sup (fg) \text{ on } [x_{i-1}, x_i]$

$m_i = \inf (fg) \text{ on } [x_{i-1}, x_i]$

$M_i' = \sup (f) \quad " \quad "$

$m_i' = \inf (f) \quad " \quad "$

$M_i'' = \sup (g) \quad " \quad "$

$m_i'' = \inf (g) \quad " \quad "$

We have $\forall x, y \in [x_{i-1}, x_i]$

$$(fg)(y) - (fg)(x) = f(y)g(y) - f(x)g(x)$$

$$+ f(x)g(y) - f(x)g(y)$$

$$= g(y)[f(y) - f(x)] + f(x)[g(y) - g(x)]$$

$$|(fg)(y) - (fg)(x)| = |g(y)[f(y) - f(x)] + f(x)[g(y) - g(x)]|$$

$$\leq |g(y)| |f(y) - f(x)| + |f(x)| |g(y) - g(x)|$$

\Rightarrow

$$M_i - m_i \leq k(M_i' - m_i') + k(M_i'' - m_i'') \rightarrow 0$$

This condition holds for all partitions of $[a, b]$

$\therefore f, g \in R[a, b]$

\therefore for given $\epsilon > 0 \quad \exists$ partitions $P_1 \neq P_2$ such

$$\text{now } M_i' - m_i' \leq \frac{M_i - m_i}{t^2} \equiv \frac{71}{t^2}$$

$$U(P, \frac{1}{f}, \alpha) - L(P, \frac{1}{f}, \alpha) \\ = \sum (M_i' - m_i') \Delta \alpha_i$$

$$\leq \frac{1}{t^2} \sum (M_i - m_i) \Delta \alpha_i$$

$$< \frac{1}{t^2} \epsilon t^2 = \epsilon$$

$$\Rightarrow \frac{1}{f} \in R(\alpha)$$

(d) If $f, g \in R(\alpha)$ and $\exists t > 0$ such that

$$|g(x)| \geq t > 0 \quad \forall x \in [a, b]$$

, then $\frac{f}{g} \in R(\alpha)$ on $[a, b]$

Proof # $\because f, g \in R(\alpha)$

$\Rightarrow f, g$ are bounded on $[a, b]$

$\Rightarrow \exists$ a no $k > 0$ such that

$$|f(x)| \leq k, \quad |g(x)| \leq k \quad \forall x \in [a, b]$$

$$\forall x \in [a, b]$$

$$\left| \left(\frac{f}{g} \right)(x) \right| = \frac{|f(x)|}{|g(x)|} \leq \frac{k}{t}$$

For any $x, y \in [x_{i-1}, x_i]$

$$\begin{aligned} \left| \left(\frac{f}{g} \right)(y) - \left(\frac{f}{g} \right)(x) \right| &= \left| \frac{f(y)}{g(y)} - \frac{f(x)}{g(x)} \right| \\ &= \left| \frac{f(y)g(x) - f(x)g(y)}{g(x)g(y)} \right| \end{aligned}$$

$$= \frac{|f(y)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(y)|}{|g(x)g(y)|}$$

$$\leq \frac{|g(x)| |f(y) - f(x)| + |f(x)| |g(x) - g(y)|}{|g(x)| |g(y)|}$$

$$\leq \frac{k}{t^2} (M_i' - m_i') + \frac{k}{t^2} (M_i'' - m_i'')$$

$$\Rightarrow M_i - m_i \leq \frac{k}{t^2} (M_i' - m_i') + \frac{k}{t^2} (M_i'' - m_i'') \rightarrow \textcircled{1}$$

$$\therefore f, g \in R(\alpha) \quad \forall P$$

$\therefore \exists$ partitions P_1, P_2 such that

$$U(P_1, f, \alpha) - L(P_1, f, \alpha) < \frac{\epsilon t^2}{2k} \rightarrow \textcircled{2}$$

$$U(P_2, g, \alpha) - L(P_2, g, \alpha) < \frac{\epsilon t^2}{2k} \rightarrow \textcircled{3}$$

Let $P = P_1 \cup P_2$, then

$$U(P, f, \alpha) - L(P, f, \alpha) < \frac{\epsilon t^2}{2k} \rightarrow \textcircled{4}$$

$$U(P, g, \alpha) - L(P, g, \alpha) < \frac{\epsilon t^2}{2k} \rightarrow \textcircled{5}$$

& For this P $\textcircled{1}$ also holds.

Now

$$U(P, \frac{f}{g}, \alpha) - L(P, \frac{f}{g}, \alpha) = \sum_{i=1}^n (M_i - m_i) \Delta \alpha_i$$

$$\leq \frac{k}{t^2} \sum (M_i' - m_i') \Delta \alpha + \frac{k}{t^2} \sum (M_i'' - m_i'') \Delta \alpha_i$$

$$< \frac{k}{t^2} \cdot \frac{\epsilon t^2}{2k} + \frac{k}{t^2} \cdot \frac{\epsilon t^2}{2k} = \epsilon$$

$$\Rightarrow \frac{f}{g} \in R(\alpha) \text{ on } [a, b]$$

5)) # Let f bounded function $[a, b]$, α_1, α_2 be defined, \uparrow on $[a, b]$, c be a true constant.

(a) # If $f \in R(\alpha_1)$, $f \in R(\alpha_2)$, then
 $f \in R(\alpha_1 + \alpha_2)$ and

$$\int_a^b f d(\alpha_1 + \alpha_2) = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2$$

(b) # If $f \in R(\alpha_1)$, then $f \in R(c\alpha_1)$ and
 $\int_a^b f d(c\alpha_1) = c \int_a^b f d\alpha_1$

Proof # $\because f \in R(\alpha_1) \& f \in R(\alpha_2)$

\therefore For $\epsilon > 0$ \exists partitions $P_1 \& P_2$

such that

$$U(P_1, f, \alpha_1) - L(P_1, f, \alpha_1) < \frac{\epsilon}{2} \rightarrow (1)$$

$$\& U(P_2, f, \alpha_2) - L(P_2, f, \alpha_2) < \frac{\epsilon}{2} \rightarrow (2)$$

$$\text{Let } P = P_1 \cup P_2$$

Then

$$U(P, f, \alpha_1) - L(P, f, \alpha_1) < \frac{\epsilon}{2} \rightarrow (3)$$

$$U(P, f, \alpha_2) - L(P, f, \alpha_2) < \frac{\epsilon}{2} \rightarrow (4)$$

$$\text{Let } M_i = \sup f \text{ on } [x_{i-1}, x_i]$$

$$m_i = \inf f \text{ " " " "}$$

$$\text{Let } \alpha = \alpha_1 + \alpha_2$$

$$\therefore \alpha(x) = \alpha_1(x) + \alpha_2(x)$$

$$\Delta \alpha_{1i} = \alpha_1(x_i) - \alpha_1(x_{i-1})$$

$$\Delta \alpha_{2i} = \alpha_2(x_i) - \alpha_2(x_{i-1})$$

$$\Delta \alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

$$= \alpha_1(x_i) + \alpha_2(x_i) - \alpha_1(x_{i-1}) - \alpha_2(x_{i-1})$$

$$= \Delta \alpha_{1i} + \Delta \alpha_{2i}$$

$$U(P, f, \alpha) = \sum M_i \Delta \alpha_i = \sum M_i \Delta \alpha_{1i} + \sum M_i \Delta \alpha_{2i}$$

$$= U(P, f, \alpha_1) + U(P, f, \alpha_2) \rightarrow (5)$$

Similarly

$$L(P, f, \alpha) = L(P, f, \alpha_1) + L(P, f, \alpha_2) \rightarrow (6)$$

$$U(P, f, \alpha) - L(P, f, \alpha)$$

$$= U(P, f, \alpha_1) + U(P, f, \alpha_2) - L(P, f, \alpha_1) - L(P, f, \alpha_2)$$

$$= U(P, f, \alpha_1) - L(P, f, \alpha_1) + U(P, f, \alpha_2) - L(P, f, \alpha_2)$$

$$< \epsilon/2 + \epsilon/2 = \epsilon \quad \text{by (2) \& (3)}$$

$$\Rightarrow f \in R(\alpha) = R(\alpha_1 + \alpha_2)$$

$$\inf_P U(P, f, \alpha) = \int_a^b f d\alpha$$

$$\leq U(P, f, \alpha)$$

$$\Rightarrow \int_a^b f d\alpha \leq U(P, f, \alpha_1) + U(P, f, \alpha_2) \quad \forall P$$

$$\int_a^b f d\alpha$$

$$\Rightarrow \int_a^b f d\alpha \leq \inf_P U(P, f, \alpha_1) + \inf_P U(P, f, \alpha_2)$$

$$= \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2$$

$$= \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2$$

$$\Rightarrow \int_a^b f d(\alpha_1 + \alpha_2) \leq \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 \rightarrow (7)$$

Again $\int_a^b f d\alpha = \sup_P L(P, f, \alpha)$

$$\geq L(P, f, \alpha) = L(P, f, \alpha_1) + L(P, f, \alpha_2)$$

$$\Rightarrow \int_a^b f d\alpha \geq L(P, f, \alpha_1) + L(P, f, \alpha_2) \quad \forall P$$

$$\int_a^b f dx \geq \sup_p L(P, f, \alpha_1) + \sup_p L(P, f, \alpha_2)$$

$$\int_a^b f dx \geq \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2$$

$$\Rightarrow \int_a^b f d(\alpha_1 + \alpha_2) \geq \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 \rightarrow (2)$$

By (7) & (8)

$$\int_a^b f d(\alpha_1 + \alpha_2) = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2$$

(b) # $\therefore f \in R(\alpha)$

\therefore for $\epsilon > 0$ \exists a partition P such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon/c \rightarrow (1)$$

For the same partition.

$$U(P, f, c\alpha) = \sum_{i=1}^n M_i \Delta(c\alpha_i)$$

$$= c \sum_{i=1}^n M_i \Delta\alpha_i$$

$$= c U(P, f, \alpha) \rightarrow (2)$$

Similarly

$$L(P, f, c\alpha) = c L(P, f, \alpha) \rightarrow (3)$$

$$U(P, f, c\alpha) - L(P, f, c\alpha)$$

$$= c U(P, f, \alpha) - c L(P, f, \alpha)$$

$$= c [U(P, f, \alpha) - L(P, f, \alpha)]$$

$$< c \cdot \epsilon/c = \epsilon$$

$\Rightarrow f \in R(c\alpha)$ on $[a, b]$

$$\therefore U(P, f, c\alpha) = c U(P, f, \alpha)$$

$$\therefore \inf_P U(P, f, c\alpha) = c \inf_P U(P, f, \alpha)$$

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$$\int_a^b f d(c\alpha) = c \int_a^b f d\alpha.$$

$$\Rightarrow \int_a^b f d(c\alpha) = c \int_a^b f d\alpha \quad \because f \in R(\alpha) \\ f \in R(c\alpha)$$

Theorem # Let α be defined and monotonic increasing on $[a, b]$. If $f \in R(\alpha)$ on $[a, b]$, $a < c < b$, then $f \in R(\alpha)$ on $[a, c]$, $f \in R(\alpha)$ on $[c, b]$ and

$$\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha.$$

Also for any sub-interval $[c, d]$ of $[a, b]$ $f \in R_\alpha[c, d]$.

Proof # $\because f \in R(\alpha)$ on $[a, b]$

$\Rightarrow f$ is bounded on $[a, b]$

$\Rightarrow f$ is bounded on $[a, c]$, $[c, b]$

$\therefore f \in R(\alpha)$ on $[a, b]$

\therefore for a given $\epsilon > 0$, \exists a partition P of $[a, b]$

such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon \longrightarrow \textcircled{1}$$

Let $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$

and $P' = \{a = x_0, x_1, \dots, x_k, c, x_{k+1}, \dots, x_n\}$

Let $P_1 = \{a = x_0, x_1, \dots, x_k, c\} = P' \cap [a, c]$

$P_2 = \{c, x_{k+1}, x_{k+2}, \dots, x_n = b\} = P' \cap [c, b]$

Then P_1 & P_2 are partitions of $[a, c]$ & $[c, b]$

such that $P' = P_1 \cup P_2$

$$\therefore p \subset p' \quad \Rightarrow \quad 77$$

$$\therefore U(P', f, \alpha) - L(P', f, \alpha) < \epsilon \quad \longrightarrow (2)$$

$$\begin{aligned} \text{Let } M_i &= \sup f \quad \text{on } [x_{i-1}, x_i] \\ m_i &= \inf f \quad \text{on } [x_{i-1}, x_i] \\ M'_k &= \sup f \quad \text{on } [x_k, c] \\ m'_k &= \inf f \quad \text{on } [x_k, c] \\ M'' &= \sup f \quad \text{on } [c, x_{k+1}] \\ m'' &= \inf f \quad \text{on } [c, x_{k+1}] \end{aligned}$$

$$\begin{aligned} U(P', f, \alpha) &= \sum_{i=1}^k M_i \Delta x_i + M'_k (c - x_k) + M'' (x_{k+1} - c) \\ &\quad + \sum_{i=k+2}^n M_i \Delta x_i \\ &= \sum_{i=1}^k M_i \Delta x_i + M'_k (c - x_k) + M'' (x_{k+1} - c) + \sum_{i=k+2}^n M_i \Delta x_i \end{aligned}$$

$$= U(P_1, f, \alpha) + U(P_2, f, \alpha) \quad \longrightarrow (3)$$

Also

$$L(P', f, \alpha) = L(P_1, f, \alpha) + L(P_2, f, \alpha) \quad \longrightarrow (4)$$

Subtracting (4) from (3)

$$[U(P_1, f, \alpha) - L(P_1, f, \alpha)] + [U(P_2, f, \alpha) - L(P_2, f, \alpha)]$$

$$= U(P', f, \alpha) - L(P', f, \alpha) < \epsilon \quad \text{by (2)} \quad \longrightarrow (5)$$

$$\therefore U(P_1, f, \alpha) - L(P_1, f, \alpha) \geq 0$$

$$\& U(P_2, f, \alpha) - L(P_2, f, \alpha) \geq 0$$

Therefore (5) \Rightarrow

$$U(P_1, f, \alpha) - L(P_1, f, \alpha) < \epsilon$$

$$\& U(P_2, f, \alpha) - L(P_2, f, \alpha) < \epsilon$$

$$\Rightarrow f \in R_\alpha[a, c] \& f \in R_\alpha[c, b]$$

Proof of $\int_a^b f dx = \int_a^c f dx + \int_c^b f dx$

$$\begin{aligned} U(P, f, \alpha) &\geq U(P', f, \alpha) = U(P_1, f, \alpha) + U(P_2, f, \alpha) \\ &\geq \int_a^c f dx + \int_c^b f dx \\ &= \int_a^c f dx + \int_c^b f dx \quad \forall P \end{aligned}$$

$$\Rightarrow \inf_P U(P, f, \alpha) \geq \int_a^c f dx + \int_c^b f dx$$

$$\Rightarrow \int_a^b f dx \geq \int_a^c f dx + \int_c^b f dx \rightarrow \textcircled{6}$$

Again

$$\begin{aligned} L(P, f, \alpha) &\leq L(P', f, \alpha) = L(P_1, f, \alpha) + L(P_2, f, \alpha) \\ &\leq \int_a^c f dx + \int_c^b f dx \\ &= \int_a^c f dx + \int_c^b f dx \end{aligned}$$

$$\Rightarrow L(P, f, \alpha) \leq \int_a^c f dx + \int_c^b f dx \quad \forall P$$

$$\Rightarrow \sup_P L(P, f, \alpha) \leq \int_a^c f dx + \int_c^b f dx$$

$$\int_a^b f dx \leq \int_a^c f dx + \int_c^b f dx \rightarrow \textcircled{7}$$

By $\textcircled{6}$ & $\textcircled{7}$

$$\int_a^b f dx = \int_a^c f dx + \int_c^b f dx$$

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Lemma # If M, m are Supremum and infimum of f and M', m' are sup. and inf of $|f|$ on $[a, b]$, Then

$$M' - m' \leq M - m$$

Proof # \therefore For any two real numbers c, d we have

$$||c| - |d|| \leq |c - d|$$

Therefore for any $x', x'' \in [a, b]$, we have

$$||f(x')| - |f(x'')|| \leq |f(x') - f(x'')| \rightarrow (A)$$

$$\therefore M = \sup f \text{ on } [a, b]$$

$$\& m = \inf f \text{ on } [a, b]$$

$$\therefore f(x) \leq M \quad \forall x \in [a, b]$$

$$f(x) \geq m \quad \forall x \in [a, b]$$

$$\therefore x', x'' \in [a, b]$$

$$\therefore f(x') \leq M \quad \& \quad f(x'') \geq m.$$

$$\rightarrow (1) \quad \Rightarrow -f(x'') \leq -m \rightarrow (2)$$

$$(1) + (2) \Rightarrow$$

$$f(x') - f(x'') \leq M - m \rightarrow (3)$$

Interchanging $x' \& x''$

$$f(x'') - f(x') \leq M - m$$

$$- [f(x') - f(x'')] \leq M - m \rightarrow (4)$$

By (3) & (4)

$$|f(x') - f(x'')| \leq M - m \rightarrow (5)$$

Using (5) in (A)

$$| |f(x')| - |f(x'')| | \leq M - m \rightarrow (6)$$

$$M' = \sup |f| \text{ on } [a, b]$$

$$m' = \inf |f| \text{ on } [a, b]$$

$$\Rightarrow |f(x)| \leq M' \text{ \& } |f(x)| \geq m' \quad \forall x \in [a, b]$$

For any $\epsilon > 0$, we can find two points

$x_1, x_2 \in [a, b]$ such that

$$|f(x_1)| > M' - \epsilon$$

$$M' - \epsilon < |f(x_1)| \rightarrow (7)$$

and $|f(x_2)| < m' + \epsilon$

$$\Rightarrow |f(x_2)| - m' < \epsilon \rightarrow (8)$$

Adding (7) & (8)

$$M' - m' - \epsilon < |f(x_1)| - |f(x_2)| + \epsilon$$

$$\Rightarrow M' - m' < |f(x_1)| - |f(x_2)| + 2\epsilon \rightarrow (9)$$

\therefore (9) holds for arbitrary ϵ and $\forall x_1, x_2 \in [a, b]$

$$\therefore M' - m' \leq |f(x_1)| - |f(x_2)| \quad \forall x_1, x_2 \in [a, b]$$

$$\Rightarrow M' - m' \leq |f(x')| - |f(x'')| \rightarrow (10)$$

Interchanging x' & x''

$$M' - m' \leq -[|f(x')| - |f(x'')|] \rightarrow (11)$$

By (10) & (11), we have

$$M' - m' \leq | |f(x')| - |f(x'')| | \rightarrow (12)$$

By (6) & (12).

$$M' - m' \leq M - m \text{ (Proved)}$$

Theorem # 8.1 If f is continuous on $[a, b]$ and α is a monotone increasing function, then $f \in R(\alpha)$ on $[a, b]$

Proof # $\because f$ is continuous on $[a, b]$

$\therefore f$ is uniformly continuous on $[a, b]$

\Rightarrow By definition of uniform continuity $\forall \epsilon > 0$
 $\exists \delta > 0$ such that

$$|f(x_1) - f(x_2)| < \frac{\epsilon}{\alpha(b) - \alpha(a)} \longrightarrow (1)$$

whenever $|x_1 - x_2| < \delta$

(The case $\alpha(a) = \alpha(b)$ is trivial)

Choose a partition $= \{a = x_0, x_1, x_2, \dots, x_n = b\}$

such that $\|P\| < \delta$. Then $\Delta x_i < \delta$

$\because f$ is continuous on $[a, b]$

$\therefore f$ is bounded on $[a, b]$ and attains its sup and on $[a, b]$

$\Rightarrow f$ attains its sup M_i inf m_i on each sub-interval

$[x_{i-1}, x_i]$

$$\Rightarrow \begin{aligned} M_i &= f(x_i'') \\ m_i &= f(x_i') \end{aligned} \quad x_i', x_i'' \in [x_{i-1}, x_i]$$

$$\because \Delta x_i < \delta \text{ \& } x_i', x_i'' \in [x_{i-1}, x_i]$$

\therefore By (1)

$$|f(x_i'') - f(x_i')| < \frac{\epsilon}{\alpha(b) - \alpha(a)}$$

$$\Rightarrow |M_i - m_i| < \frac{\epsilon}{\alpha(b) - \alpha(a)}$$

$$\Rightarrow M_i - m_i < \frac{\epsilon}{\alpha(b) - \alpha(a)} \longrightarrow (2)$$

$$U(P, f, \alpha) - L(P, f, \alpha)$$

$$= \sum M_i \Delta \alpha_i - \sum m_i \Delta \alpha_i$$

$$= \sum_{i=1}^n (M_i - m_i) \Delta \alpha_i$$

$$< \frac{\epsilon}{\alpha(b) - \alpha(a)} \sum_{i=1}^n \Delta \alpha_i$$

$$= \frac{\epsilon}{\alpha(b) - \alpha(a)} [\alpha(b) - \alpha(a)] = \epsilon$$

$$\Rightarrow U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

$$\Rightarrow f \in R(\alpha) \text{ on } [a, b]$$

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Theorem # If f is monotone on $[a, b]$ and α is continuous and monotone increasing on $[a, b]$, then $f \in R(\alpha)$

Proof # Let f be monotonically increasing on $[a, b]$, Then

$$f(a) \leq f(x) \leq f(b) \quad \forall x \in [a, b]$$

$\Rightarrow f$ is bounded on $[a, b]$ and $\inf f = f(a)$ and $\sup f = f(b)$. Let $\epsilon > 0$ be given.

Let $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ be a partition $[a, b]$

$\because f$ is monotone increasing

$$\therefore M_i = \sup f \text{ on } [x_{i-1}, x_i] = f(x_i)$$

$$m_i = \inf f \text{ on } [x_{i-1}, x_i] = f(x_{i-1})$$

Q3

$\therefore \alpha$ is continuous and monotone increasing on the closed interval $[a, b]$

$\therefore \alpha$ assumes every value between bounds $\alpha(a), \alpha(b)$

Therefore we can choose Δx_i such that each Δx_i is same. i.e

$$\Delta x_i = \frac{\alpha(b) - \alpha(a)}{n} \quad \text{Where } n \text{ is the integer}$$

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^n (M_i - m_i) \Delta x_i$$

$$= \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^n (M_i - m_i)$$

$$= \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^n [f(x_i) - f(x_{i-1})]$$

$$= \frac{\alpha(b) - \alpha(a)}{n} [f(b) - f(a)]$$

$\angle \epsilon$ for large n

$$\text{Now } \frac{\alpha(b) - \alpha(a)}{n} [f(b) - f(a)] < \epsilon \rightarrow \textcircled{1}$$

$$\Rightarrow \frac{\alpha(b) - \alpha(a)}{n} [f(b) - f(a)] < n$$

$$n \cdot n > \frac{\alpha(b) - \alpha(a)}{\epsilon} [f(b) - f(a)] \rightarrow \textcircled{2}$$

\Rightarrow for a given $\epsilon > 0$ we can choose.

no n of points of division in partition

which satisfies $\textcircled{1}$ & hence $\textcircled{1}$

So for every $\epsilon > 0$, we can choose a partition

P such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

$$\Rightarrow f \in R(\alpha) \text{ on } [a, b]$$

Note $f \in R(\alpha)$ i.e. $\int_a^b f d\alpha$ exists when either

- (i) f is continuous & α is monotone or
- (ii) f is monotone & α is continuous & monotone

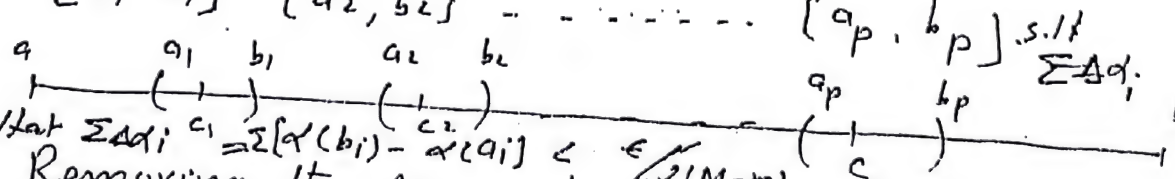
Theorem # Let f bounded on $[a, b]$, f has only finitely many points of discontinuity on $[a, b]$ and α is continuous at every point at which f is continuous. Then $f \in R(\alpha)$

Proof Let c_1, c_2, \dots, c_p be finite no of points of discontinuity of f on $[a, b]$ such that $c_1 < c_2 < c_3 < \dots < c_p$. Let $\epsilon > 0$

\therefore Set E of these points is finite

\therefore we can enclose the points of E in the interior of finitely many disjoint intervals

$[a_1, b_1], [a_2, b_2], \dots, [a_p, b_p]$ s.t. $\sum \Delta \alpha_i < \epsilon$



that $\sum \Delta \alpha_i = \sum [\alpha(b_i) - \alpha(a_i)] < \epsilon$

Removing the segments (a_i, b_i) from $[a, b]$, we get a set K of remaining points i.e. intervals $[a, a_1], [b_1, a_2], \dots, [b_{p-1}, a_p], [b_p, b]$

$\therefore f$ is continuous on these intervals

$\therefore f$ is uniformly continuous on all these intervals i.e. on set K .

Let $\epsilon > 0$. $\exists \delta > 0$ such that

$$|f(s) - f(t)| < \epsilon \quad \forall s, t \in K, |s - t| < \delta$$

Also we make disjoint intervals $[a_1, b_1], [a_2, b_2], \dots, [a_p, b_p]$ in such way $\rightarrow \textcircled{1}$

that

$$\sum \Delta d_i < \frac{\epsilon}{2(M-m)} \quad \text{where } M, m \text{ are}$$

sup & inf of f on $[a, b]$

We form a partition $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ such that P contains d_i, b_i but contains no point of any segment (d_i, b_i)

and if x_{i-1} is not one of d_i , then $\Delta x_i < \delta$

Now if x_{i-1} is not one of d_i i.e. if $x_{i-1} \in K$, then.

$$\Delta x_i = x_i - x_{i-1} = |x_i - x_{i-1}| < \delta$$

for such intervals, we have

$$M_i - m_i < \frac{\epsilon}{2[\alpha(b) - \alpha(a)]}$$

If x_{i-1} is one of points d_i , then other ^{end} point will be some b_i and for such intervals ① is not true even though distance b/w points is less than δ because points of these intervals are not in K . For such intervals, we have.

$$m \leq m_i \leq M_i \leq M$$

$$\Rightarrow M_i - m_i \leq M - m$$

Let A be the set of indices for which 1st end point is not one of d_i & B be the set of those indices for which $[x_{i-1}, x_i]$ has some d_i as 1st end point.

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i \in A} (M_i - m_i) \Delta d_i + \sum_{i \in B} (M_i - m_i) \Delta d_i$$

$$< \frac{\epsilon}{2[\alpha(b) - \alpha(a)]} \sum \Delta d_i + (M - m) \sum \Delta d_i$$

$$\leq \frac{\epsilon}{2[\alpha(b) - \alpha(a)]} [\alpha(b) - \alpha(a)] + (M - m) \cdot \frac{\epsilon}{2(M - m)} = \epsilon$$

$$\Rightarrow f \in R(\alpha)$$

برائے فروغ و تعلیم

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Integrability of ⁸⁶Composition of Function#

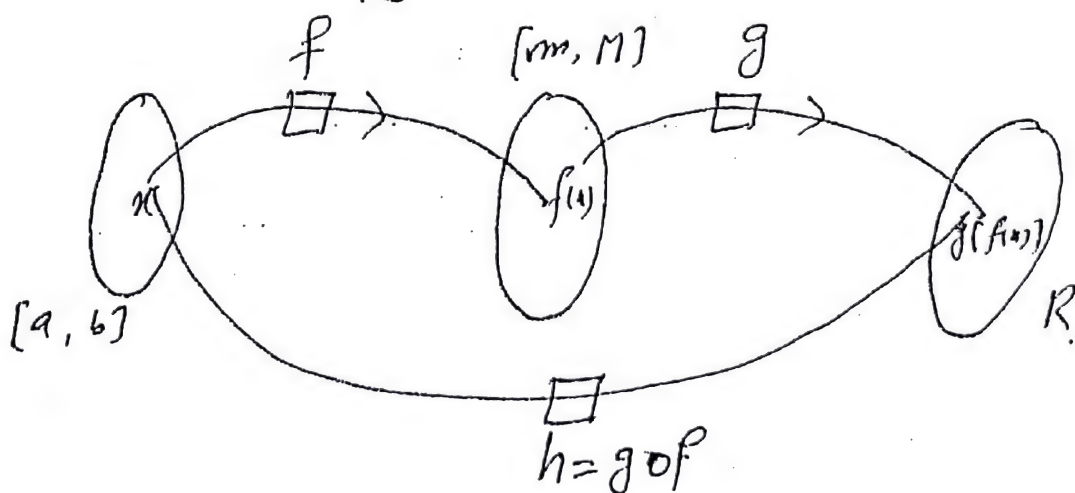
Theorem# Let f be a bounded function on $[a, b]$ and let α be monotone increasing on $[a, b]$. Suppose that $m \leq f(x) \leq M$ $\forall x \in [a, b]$. If $f \in R(\alpha)$ on $[a, b]$ and g is continuous on $[m, M]$, then $h = g \circ f \in R_\alpha[a, b]$

Proof# Let $h = g \circ f$

\because A function continuous on a closed interval is bounded and g is continuous on $[m, M]$

$\therefore g$ is bounded on $[m, M]$

Let $|g(t)| \leq K \quad \forall t \in [m, M]$



$\because g$ is continuous on $[m, M]$

$\therefore g$ is uniformly continuous on $[m, M]$

Therefore for $\epsilon > 0 \quad \exists \delta, 0 < \delta < \frac{\epsilon}{2K + \alpha(b) - \alpha(a)}$

such that

$$|g(s) - g(t)| < \frac{\epsilon}{2K + \alpha(b) - \alpha(a)} \quad \text{if } |s - t| < \delta \quad \text{--- } \textcircled{1}$$

$\because f \in R(\alpha)$

$\therefore \exists$ a partition $P = \{a, x_0, x_1, x_2, \dots, x_n, b\}$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \delta^2 \rightarrow \textcircled{2}$$

$$\text{Let } M_i = \sup f \text{ on } [x_{i-1}, x_i]$$

$$m_i = \inf f \text{ " " "}$$

$$M_i' = \sup f \text{ " " "}$$

$$m_i' = \inf f \text{ " " "}$$

Dividing the indices $i=1, 2, 3, \dots, n$ into two classes A & B as under

$$A = \{i \mid 1 \leq i \leq n \text{ \& } M_i - m_i < \delta\}$$

$$B = \{i \mid 1 \leq i \leq n \text{ \& } M_i - m_i \geq \delta\}$$

if $i \in A$, we have by $\textcircled{1}$

$$M_i' - m_i' < \frac{\epsilon}{2K + \alpha(b) - \alpha(a)}$$

if $i \in B$, we have from

$$|g(t)| \leq K \quad \forall t \in [a, b]$$

$$\Rightarrow -K \leq g(t) \leq K.$$

$$\Rightarrow 0 \leq g(t) \leq 2K$$

$$\Rightarrow M_i' - m_i' \leq 2K.$$

$$\text{Now } \delta \leq M_i - m_i \quad \forall i \in B$$

$$\sum_{i \in B} \delta \Delta \alpha_i \leq \sum_{i \in B} (M_i - m_i) \Delta \alpha_i \leq \sum_{i=1}^n (M_i - m_i) \Delta \alpha_i$$

$$= U(P, f, \alpha) - L(P, f, \alpha) < \delta^2 \text{ by } \textcircled{2}$$

$$\Rightarrow \delta \sum_{i \in B} \Delta \alpha_i < \delta^2 \Rightarrow \sum_{i \in B} \Delta \alpha_i < \delta \rightarrow \textcircled{3}$$

Therefore

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^n (M_i' - m_i') \Delta \alpha_i$$

$$= \sum_{i \in A} (M_i' - m_i') \Delta \alpha_i + \sum_{i \in B} (M_i' - m_i') \Delta \alpha_i$$

$$< \frac{\epsilon}{2K + \alpha(b) - \alpha(a)} \sum_{i \in A} \Delta \alpha_i + 2K \sum_{i \in B} \Delta \alpha_i$$

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$$< \frac{\epsilon}{2K + \alpha(b) - \alpha(a)} \sum_{i \in A}^{\frac{88}{\epsilon}} \Delta \alpha_i + 2K\delta$$

$$< \frac{\epsilon}{2K + \alpha(b) - \alpha(a)} \sum_{i=1}^n \Delta \alpha_i + 2K\delta \quad \because \sum_{i \in A} \Delta \alpha_i \leq \sum_{i=1}^n \Delta \alpha_i$$

$$= \frac{\epsilon}{2K + \alpha(b) - \alpha(a)} [\alpha(b) - \alpha(a)] + 2K \frac{\epsilon}{2K + \alpha(b) - \alpha(a)}$$

$$= \frac{\epsilon [2K + \alpha(b) - \alpha(a)]}{2K + \alpha(b) - \alpha(a)} = \epsilon$$

$$\Rightarrow U(P, h, \alpha) - L(P, h, \alpha) < \epsilon$$

$$\Rightarrow h = g \circ f \in R(\alpha)$$

Reduction To Riemann Integral

Theorem # Assume α increases monotonically and $\alpha' \in R$ on $[a, b]$. Let f be bounded function on $[a, b]$. Then $f \in R(\alpha)$ iff $f\alpha' \in R$ on $[a, b]$ and $\int_a^b f d\alpha = \int_a^b f(x) \alpha'(x) dx$

Proof # $\because \alpha' \in R$ on $[a, b]$

\therefore For a given $\epsilon > 0 \exists$ a partition

$P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ of $[a, b]$ s.t

$$U(P, \alpha') - L(P, \alpha') < \epsilon \longrightarrow \textcircled{1}$$

$\therefore \alpha'$ exists on $[a, b]$

$\therefore \alpha$ is continuous on $[a, b]$ & hence on $[x_{i-1}, x_i]$

and its derivative also exists in $]x_{i-1}, x_i[$

By MVT There exists points $t_i \in]x_{i-1}, x_i[$

such that

$$\frac{\alpha(x_i) - \alpha(x_{i-1})}{x_i - x_{i-1}} = \alpha'(t_i) \quad \forall i$$

$$\Rightarrow \alpha(x_i) - \alpha(x_{i-1}) = (x_i - x_{i-1}) \alpha'(t_i) \quad \forall i$$

$$\Rightarrow \Delta \alpha_i = \Delta x_i \alpha'(t_i) \quad \forall i$$

If $x_i \in [x_{i-1}, x_i]$, then $\alpha'(t_i), \alpha'(x_i)$ both lie in $[m_i, M_i]$, where

$$M_i = \sup \alpha' \text{ on } [x_{i-1}, x_i]$$

$$m_i = \inf \alpha' \text{ on } [x_{i-1}, x_i]$$

$$\Rightarrow |\alpha'(x_i) - \alpha'(t_i)| \leq M_i - m_i$$

$$\Rightarrow \sum_{i=1}^n |\alpha'(x_i) - \alpha'(t_i)| \Delta x_i \leq \sum_{i=1}^n (M_i - m_i) \Delta x_i$$

$$< \epsilon \quad \text{by (1)}$$

$$\Rightarrow \sum_{i=1}^n |\alpha'(x_i) - \alpha'(t_i)| \Delta x_i < \epsilon \quad \rightarrow (2)$$

$\therefore f$ is bounded on $[a, b]$

$$\text{Let } |f(x)| \leq M \quad \text{where } M > 0 \\ \forall x \in [a, b]$$

Step-II

Consider

$$\left| \sum_{i=1}^n f(x_i) \Delta \alpha_i - \sum_{i=1}^n f(x_i) \alpha'(x_i) \Delta x_i \right|$$

$$= \left| \sum_{i=1}^n f(x_i) \Delta \alpha_i - \sum_{i=1}^n f(x_i) \alpha'(x_i) \Delta x_i \right|$$

$$= \left| \sum_{i=1}^n f(x_i) \alpha'(t_i) \Delta x_i - \sum_{i=1}^n f(x_i) \alpha'(x_i) \Delta x_i \right|$$

$$\because \Delta \alpha_i = \alpha'(x_i) \Delta x_i$$

$$= \left| \sum_{i=1}^n f(x_i) [\alpha'(t_i) - \alpha'(x_i)] \Delta x_i \right|$$

$$\leq \left| \sum_{i=1}^n M [\alpha'(t_i) - \alpha'(x_i)] \Delta x_i \right|$$

$$\leq M \epsilon$$

by (2)

$\rightarrow (3)$

$$\Rightarrow -M\epsilon < \sum f(x_i) \Delta x_i - \sum f(x_i) \alpha'(x_i) \Delta x_i < M\epsilon$$

$$\Rightarrow \sum_{i=1}^n f(x_i) \Delta x_i < \sum_{i=1}^n f(x_i) \alpha'(x_i) \Delta x_i + M\epsilon \rightarrow (5)$$

$\forall x_i \in [x_{i-1}, x_i]$

$$\text{and } \sum_{i=1}^n f(x_i) \alpha'(x_i) \Delta x_i < \sum_{i=1}^n f(x_i) \Delta x_i + M\epsilon \rightarrow (6)$$

$\forall x_i \in [x_{i-1}, x_i]$

$$\Rightarrow U(P, f, \alpha) \leq U(P, f, \alpha') + M\epsilon \rightarrow (7)$$

$$\text{and } U(P, f, \alpha') \leq U(P, f, \alpha) + M\epsilon \rightarrow (8)$$

from (7) & (8) we have.

$$|U(P, f, \alpha) - U(P, f, \alpha')| \leq \epsilon M \rightarrow (9)$$

This remains true if P is replaced by any refinement because (1) remains true for any refinement of P

$$\text{Therefore, } \overline{a} \quad \overline{b}$$

$$\left| \int_a^b f d\alpha - \int_a^b f(x) \alpha'(x) dx \right| \leq M\epsilon$$

$\therefore \epsilon$ is an arbitrary and may be very small. Therefore.

~~Similarly from (4)~~

$$\left| \int_a^b f d\alpha - \int_a^b f(x) \alpha'(x) dx \right| = 0$$

$$\Rightarrow \int_a^b f d\alpha = \int_a^b f(x) \alpha'(x) dx \rightarrow (10)$$

Similarly from (4) we can prove that

$$\int_a^b f d\alpha = \int_a^b f(x) \alpha'(x) dx \rightarrow (11)$$

Hence if $f \in R(\alpha)$ and $[a, b]$, then

$$\int_a^b f d\alpha = \int_a^b f d\alpha = \int_a^b f d\alpha.$$

and from (i) & (ii) we have.

$$\int_a^b f(u) \alpha'(u) du = \int_a^b f(u) \alpha'(u) du$$

$$\Rightarrow f \alpha' \in R \text{ on } [a, b]$$

Conversely if $f \alpha' \in R$, then from (i) & (ii)

$$\int_a^b f d\alpha = \int_a^b f d\alpha$$

$$\Rightarrow f \in R(\alpha) \text{ on } [a, b]$$

Change of Variable in Riemann.

Stieltjes Integral

Theorem # Let $f \in R_\alpha[a, b]$ and ϕ be a strictly increasing continuous function that maps an interval $[A, B]$ onto $[a, b]$. Suppose α is monotonically increasing on $[a, b]$.

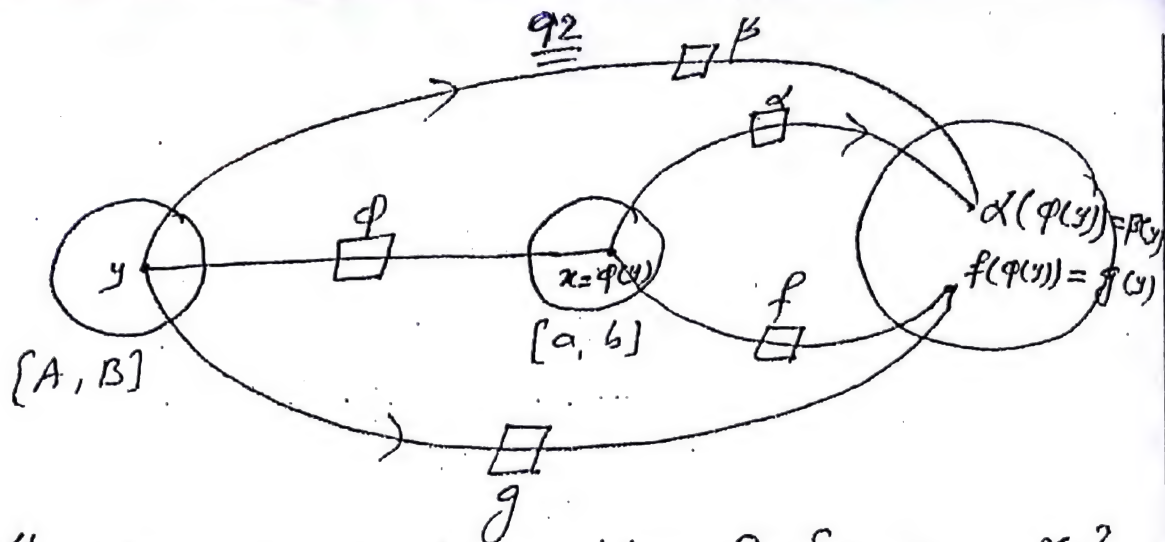
Let β & g be composite function defined on $[A, B]$ as

$$\begin{aligned} \beta(y) &= \alpha[\phi(y)] \\ g(y) &= f[\phi(y)] \quad \forall y \in [A, B] \end{aligned}$$

Then $g \in R(\beta)$ on $[A, B]$ and

$$\int_A^B g d\beta = \int_a^b f d\alpha.$$

Proof # $\because \phi$ is strictly increasing on $[A, B]$
 $\therefore \phi$ is 1-1 and has strictly increasing inverse ϕ^{-1} on $[a, b]$



Therefore to each partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$, there corresponds one and only one partition $Q = \{y_0, y_1, \dots, y_n\}$ of $[A, B]$ so that

$$x_i = \phi(y_i)$$

All partitions of $[a, b]$ are obtained in this way

Also any refinement of Q produces a corresponding refinement of P and converse is also true.

\therefore The values taken by f on $[x_{i-1}, x_i]$ are exactly same as those taken by g on $[y_{i-1}, y_i]$. So

$$U(Q, g, \beta) = U(P, f, \alpha) \rightarrow (1)$$

$$L(Q, g, \beta) = L(P, f, \alpha) \rightarrow (2)$$

$\therefore f \in R(\alpha)$ on $[a, b]$

\therefore given $\epsilon > 0$, we have.

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

$$\Rightarrow U(Q, g, \beta) - L(Q, g, \beta) < \epsilon$$

$\Rightarrow g \in R(\beta)$ on $[A, B]$

$$\text{from (1)} \quad \inf_Q U(Q, g, \beta) = \inf_P U(P, f, \alpha)$$

$$\int_A^B g d\beta = \int_a^b f d\alpha = \int_a^b f d\alpha$$

Remarks # This ⁹³Theorem applies in particular to Riemann integrals i.e. when $\alpha(x) = x$. Another Theorem of this type in which φ is not required to be monotone will later be proved for Riemann integral.

Question # Suppose that $f \geq 0$ i.e. f is non-negative, f is continuous on $[a, b]$ and $\int_a^b f(x) dx = 0$. Prove that $f(x) = 0$ $\forall x \in [a, b]$. Does its converse hold. Prove your assertion.

Sol # Since f is continuous on $[a, b]$, it is integrable. Suppose that there a point x_0 in $[a, b]$ where $f(x_0) > 0$ i.e. $f(x_0) \neq 0$.

Now if a function f is continuous at point c and $f(c) \neq 0$, then f is locally bounded away from zero i.e. \exists a nbd $N(c)$ of c and the constant m such that $|f(x)| \geq m > 0 \quad \forall x \in [a, b] \cap N(c)$.

$\therefore f$ is continuous at x_0 & $f(x_0) \neq 0$

$\therefore f$ is bounded away from zero at x_0 i.e. \exists a no $K > 0$ and nbd $N(x_0)$ of x_0 such that $0 < m \leq |f(x)| = f(x) \quad \therefore f(x) \geq 0$ $\forall x \in N(x_0) \cap [a, b]$

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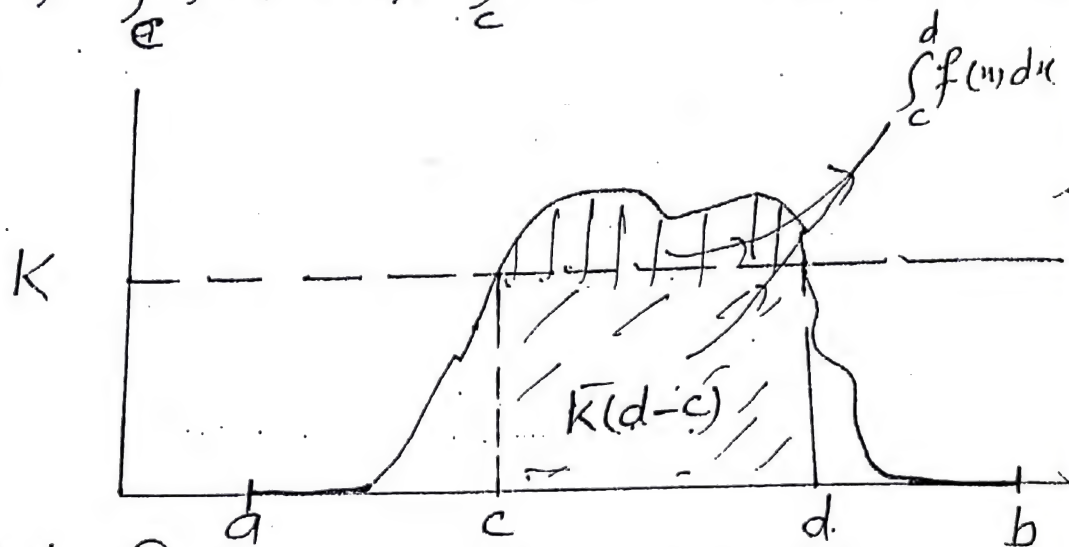
Let the interval $[a, b] \cap N(x_0)$ have end points c, d with $c < d$

$\therefore f$ is continuous on $[c, d]$

$\therefore f$ is integrable on $[c, d]$

Now $\int_c^d f(x) dx \geq \int_c^d K dx$ on $[c, d]$

$$\Rightarrow \int_c^d f(x) dx \geq \int_c^d K dx = K(d-c) > 0$$



But $f(x) \geq 0 \quad \forall x \in [a, b]$

Therefore,

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^d f(x) dx + \int_d^b f(x) dx \geq \int_c^d f(x) dx > 0$$

$\Rightarrow \int_a^b f(x) dx > 0$ which contradicts the hypothesis that $\int_a^b f(x) dx = 0$

Therefore there can exist no point in $[a, b]$ where f is not zero. We conclude that f is identically zero on $[a, b]$

Converse ⁹⁵

Its Converse is also true i.e. if $f(x) \geq 0$ & $f(x) = 0, \forall x \in [a, b]$, then

$$\int_a^b f dx = 0$$

$$\therefore f(x) \geq 0 \text{ \& } f(x) = 0$$

$$\therefore U(P, f) = 0 = L(P, f) \quad \forall P$$

Therefore $\int_a^b f dx = 0$

By: M. Hussain Assistant Professor Asghar Mall College

Integration & Differentiation

Function Defined by Definite Integrals

Let f be Riemann integrable on $[a, b]$, Then The function F given by

$$F(x) = \int_a^x f(t) dt \quad \forall x \in [a, b]$$

is well defined. because for each $x \in [a, b]$

$f \in R[a, x]$ and as such $F(x)$ is uniquely defined on $[a, b]$. The function F may be called the integral function of f .

We examine certain properties of this function F defined on $[a, b]$

Note we observe that $F(a) = \int_a^a f dt$ and $F(b) = \int_a^b f dt$.

Theorem# (Indefinite-integral Theorem
OR
1st fundamental Theorem)

Let $f \in R[a, b]$ (or let f be continuous on $[a, b]$). Then function F defined by

$$F(x) = \int_a^x f(t) dt \quad \forall x \in [a, b]$$

is continuous on $[a, b]$; further if f is continuous at a point x_0 of $[a, b]$, then F is differentiable at x_0 and

$$F'(x_0) = f(x_0)$$

Proof# $\because f \in R[a, b]$

$\therefore f$ is bounded on $[a, b]$

Let $|f(t)| \leq M \quad \forall t \in [a, b]$

Let us choose any two points in $[a, b]$ such that

$$a \leq x < y \leq b$$

$$F(y) - F(x) = \int_a^y f(t) dt - \int_a^x f(t) dt$$

$$= \int_a^y f(t) dt + \int_x^a f(t) dt$$

$$= \int_x^y f(t) dt + \int_a^x f(t) dt$$

$$= \int_x^y f(t) dt$$

$$\Rightarrow |F(y) - F(x)| = \left| \int_x^y f(t) dt \right| \leq M(y-x)$$

For any $\epsilon > 0$ such that
 $M|y-x| < \epsilon$ when $|y-x| < \frac{\epsilon}{M}$

Then $|F(y) - F(x)| < \epsilon$ whenever $|y-x| < \frac{\epsilon}{M} = \delta$

so that for any $\epsilon > 0$ we can always
 choose $\delta = \frac{\epsilon}{M}$ such that

$|F(y) - F(x)| < \epsilon$ whenever $|y-x| < \delta$

$\Rightarrow F$ is uniformly continuous on $[a, b]$
 \therefore uniformly continuous function is

continuous

$\therefore F$ is continuous on $[a, b]$

Differentiability At x_0

Suppose that f is continuous at x_0 .

Then by definition of continuity for $\epsilon > 0$

$\exists \delta > 0$ such that

$|f(t) - f(x_0)| < \epsilon$ whenever $|t - x_0| < \delta$
 $\rightarrow \textcircled{1}$

$\Rightarrow f(x_0) - \epsilon < f(t) < f(x_0) + \epsilon$ $x_0 - \delta < t < x_0 + \delta$

Now

$$\frac{F(t) - F(x_0)}{t - x_0} - f(x_0)$$

$$= \frac{1}{t - x_0} \int_{x_0}^t f(t) dt - f(x_0) \quad \text{by definition of } F$$

$$= \frac{1}{t - x_0} \left[\int_{x_0}^t f(t) dt - (t - x_0) f(x_0) \right]$$

$$= \frac{1}{t-x_0} \left\{ \int_{x_0}^t f(t) dt - f(x_0) \int_{x_0}^t dt \right\}$$

$$\Rightarrow \left| \frac{F(t) - F(x_0)}{t - x_0} - f(x_0) \right| = \frac{1}{|t - x_0|} \left| \int_{x_0}^t (f(t) - f(x_0)) dt \right|$$

$$\leq \frac{1}{|t - x_0|} \int_{x_0}^t |f(t) - f(x_0)| dt \longrightarrow (2)$$

From (1)

$$\int_{x_0}^t |f(t) - f(x_0)| dt < \int_{x_0}^t \epsilon dt \quad \text{when } |t - x_0| < \delta$$

$$\int_{x_0}^t |f(t) - f(x_0)| dt < \epsilon |t - x_0| \quad |t - x_0| < \delta$$

Using in (2)

$$\left| \frac{F(t) - F(x_0)}{t - x_0} - f(x_0) \right| < \frac{1}{|t - x_0|} \epsilon |t - x_0| = \epsilon$$

$$\Rightarrow \left| \frac{F(t) - F(x_0)}{t - x_0} - f(x_0) \right| < \epsilon \quad \text{provided } 0 < |t - x_0| < \delta$$

Hence we conclude that F is differentiable at x_0 and

$$\lim_{t \rightarrow x_0} \frac{F(t) - F(x_0)}{t - x_0} = f(x_0)$$

$$\Rightarrow F'(x_0) = f(x_0)$$

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Results # (1) we note that function.

$F(u) = \int_a^u f(t) dt$ is always continuous though $f(u)$ may not be continuous. Thus the process of integration generates continuous functions.

(2) # If f is continuous on $[a, b]$, then F is differentiable on $[a, b]$. Thus the process of integration applied to continuous functions generates differentiable functions. If f is differentiable at a point, then F is twice differentiable at that point.

(3) Every continuous function f on $[a, b]$ is derivative of a function

$$F(u) = \int_a^u f(t) dt$$

$$\& \frac{d}{du} [F(u)] = f(u)$$

$\Rightarrow F$ is a member of family of primitives of f on $[a, b]$.

The family of primitives of a continuous function f is denoted by $\int f dx$ and is called indefinite integral of f .

Primitive # if f and F are two functions defined on $[a, b]$ and $F'(u) = f(u) \quad \forall u \in [a, b]$, then F is called a primitive of f on $[a, b]$.

Theorem # Suppose that F be a primitive of a continuous function f on $[a, b]$. A function G defined on $[a, b]$ is also a primitive of f on $[a, b]$ iff for some constant C , $G(x) = F(x) + C$ $\forall x \in [a, b]$

Proof # Suppose G is a primitive of f . Then $F'(x) = f(x) \quad \forall x \in [a, b]$

$$G'(x) = f(x) \quad \forall x \in [a, b]$$

$$\Leftrightarrow F'(x) = G'(x)$$

$$\Leftrightarrow F'(x) - G'(x) = 0$$

$$\Leftrightarrow (F' - G')(x) = 0$$

$$\Leftrightarrow F - G = C$$

$$\Leftrightarrow F = G + C$$

-or

$$G = F - C = F + C_1$$

Note (1) Any two primitives of a function differ by a constant.

(2) Primitive is also called anti-derivative.

Cauchy's Fundamental Theorem of Calculus 101 OR

2nd Fundamental Theorem

If $f \in R[a, b]$ and there is a differentiable function F on $[a, b]$ such that $F'(u) = f(u) \quad \forall u \in [a, b]$, then

$$\int_a^b f(u) du = F(b) - F(a)$$

OR
If f is continuous on $[a, b]$ and F is a primitive of f on $[a, b]$, then

$$\int_a^b f du = F(b) - F(a)$$

Proof $\because f \in R[a, b]$

\therefore For any $\epsilon > 0$, \exists a partition

$$P = \{a = x_0, x_1, x_2, \dots, x_n = b\} \text{ of } [a, b]$$

such that

$$U(P, f) - L(P, f) < \epsilon \quad (1)$$

$\because F$ is differentiable on $[a, b]$

\therefore By MVT \exists points $t_i \in]x_{i-1}, x_i[$

such that

$$\frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = F'(t_i) = f(t_i) \quad \forall i = 1, 2, \dots, n$$

$$\Rightarrow F(x_i) - F(x_{i-1}) = f(t_i) \Delta x_i \quad \forall i$$

Summing over all the intervals 102

$$\sum_{i=1}^n [F(x_i) - F(x_{i-1})] = \sum_{i=1}^n f(t_i) \Delta x_i$$

$$F(b) - F(a) = \sum_{i=1}^n f(t_i) \Delta x_i$$

(telescoping sum)

This will be true for every partition of $[a, b]$

Now under condition ① for any t_i in $[x_{i-1}, x_i]$ we have

$$\left| \sum_{i=1}^n f(t_i) \Delta x_i - \int_a^b f dx \right| < \epsilon$$

$$\Rightarrow \left| F(b) - F(a) - \int_a^b f dx \right| < \epsilon$$

Since it holds for any arbitrary $\epsilon > 0$, therefore

$$\int_a^b f dx = F(b) - F(a)$$

Corollary # If f is differentiable on $[a, b]$ and if f' is integrable on $[a, b]$, then

$$\int_a^b f'(u) du = f(b) - f(a)$$

Actually we can write

$$\int_a^x f'(u) du = f(x) - f(a)$$

Note # The fundamental theorem does not state that if f is integrable, then f has a primitive. It only states that if f has

a primitive on $[a, b]$, then the primitive can be used to evaluate $\int_a^b f(x) dx$. The question whether a primitive exists and the question of the existence of an integral of function $f(x)$ on $[a, b]$ are entirely independent.

Counter Example

A function may have a primitive without being integrable.

Consider the function F and f defined on $[-1, 1]$ as

$$F(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$\text{and } f(x) = \begin{cases} 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

clearly $F'(x) = f(x)$

$\Rightarrow F$ is a primitive of f on $[-1, 1]$

But f is not integrable on $[-1, 1]$ because f is not bounded on $[a, b]$

Change of Variable in a Riemann

Integral

The formula $\int_A^B g d\beta = \int_a^b f dx$ assumes the form previously proved

$$\int_a^b f(u) du = \int_A^B f[g(t)] g'(t) dt$$

When $\alpha(x) = x$ & g is a strictly monotone function with a continuous derivative. It is valid if $f \in R[a, b]$. When f is continuous we remove the restriction that g is monotone.

Theorem # (Change of Variable)

Suppose that u has continuous derivative on $[c, d]$. Let f be continuous on the range of u . Let $u(c) = a$, $u(d) = b$. Then

$$\int_a^b f(u) du = \int_c^d f[u(t)] u'(t) dt$$

Let u be OR continuously differentiable function on $I = [c, d]$. Let $u(c) = a$, $u(d) = b$. Let f be continuous on the $u(I)$ i.e. range of u , then

$$\int_a^b f(u) du = \int_c^d f[u(t)] u'(t) dt$$

Proof \therefore The range of a continuous function

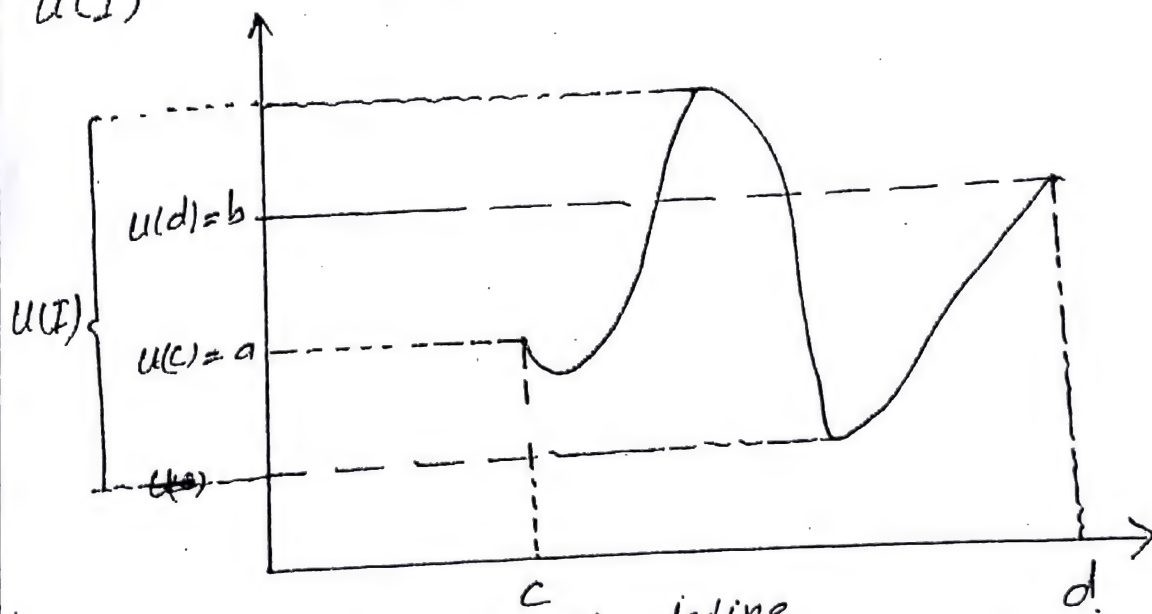
on a closed and bounded is also a closed and bounded interval
 $\therefore u$ has a range which is also a closed and interval

Since Composite function of two continuous function is also continuous, therefore $f(u)$ is continuous on $[c, d]$ and hence.

integrable because ¹⁰⁵ Continuous function is integrable.

Again $(f \circ u)u' = f(u)u'$ being a product of two continuous functions is continuous on $[c, d]$

Note that u is not assumed to be monotone so that $u(I)$ contains, but may not equal, the interval with end points $u(c) = a$ & $u(d) = b$. However since f is assumed continuous on the interval $u(I)$, it is integrable on every sub-interval of $u(I)$. See fig



For all $x \in u(I)$, define.

$$F(x) = \int_a^x f(t) dt \longrightarrow (1)$$

Also define G on $[c, d]$ as follows

$$G(x) = \int_c^x f[u(t)] u'(t) dt \longrightarrow (2)$$

We show that

$$F(u(x)) = G(x)$$

$\therefore f(u)u'$ is continuous

\therefore By property of ¹⁰⁶ integral function.
derivative of G is equal to $f(u)u'$ on
[c, d] i.e

$$G'(u) = f[u(u)]u'(u) \quad \forall u \in [c, d] \rightarrow (3)$$

But by chain rule.

$$(F \circ u)'(u) = F'[u(u)]u'(u) = f[u(u)]u'(u) \quad \forall u \in [c, d] \rightarrow (4)$$

By (3) & (4)

$$G'(u) = F'[u(u)]$$

$$\therefore F'(u) = f(u) \quad \therefore G'(u) = f(u)$$

$\therefore G(u)$ & $F[u(u)]$ are two primitives of f
on [c, d]

But any two primitives differ by a constant
D such that

$$F(u(u)) = G(u) + D \rightarrow (5)$$

$$\Rightarrow F[u(u)] = \int_a^u f(t) dt = \int_c^u f[u(t)]u'(t) dt + D$$

Let $u=c$ so that $u(c)=a$, we obtain.

$$\int_a^a f(t) dt = \int_c^c f[u(t)]u'(t) dt + D$$

$$0 = 0 + D \Rightarrow D = 0$$

By (5)

$$F[u(u)] = G(u) \quad \forall u \in [c, d]$$

$$\text{So } F[u(d)] = G(d)$$

$$\Rightarrow F(b) = G(d)$$

$$\Rightarrow \int_a^b f(t) dt = \int_c^d f[u(t)]u'(t) dt$$

OR

$$\int_a^b f(u) du = \int_c^d f[u(t)]u'(t) dt$$

Integration by Parts#

Theorem# If F and G are primitives of f and g respectively and f, g are Riemann-integrable functions, then

$$\int_a^b F(x) g(x) dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x) G(x) dx$$

Proof# \because Primitive of a function is continuous

$\therefore F, G$ are continuous on $[a, b]$

Again a continuous function on $[a, b]$ is Riemann integrable on $[a, b]$, therefore F, G are Riemann integrable on $[a, b]$.

\therefore Product of two Riemann integrable on $[a, b]$ is Riemann integrable

$\therefore Fg$ & fG are integrable on $[a, b]$

\Rightarrow The two integrals in the theorem exist

$$\text{Let } H(x) = F(x)G(x)$$

$$H'(x) = F'(x)G(x) + F(x)G'(x)$$

$$= f(x)G(x) + F(x)g(x) = h(x) \text{ say}$$

$\therefore F, G$ are integrable

$\therefore H = FG$ is also integrable.

Also fG, Fg are integrable and $h = fG + Fg$ is integrable on $[a, b]$

$$\therefore H'(x) = h(x)$$

$\therefore H(x)$ is a primitive of $h(x)$

By fundamental theorem of Calculus

$$\int_a^b h(x) dx = H(b) - H(a)$$

$$\begin{aligned}
 &\Rightarrow \int_a^b [f(u)G(u) + F(u)g(u)]du = H(b) - H(a) \\
 &\Rightarrow \int_a^b f(u)G(u)du + \int_a^b F(u)g(u)du = H(b) - H(a) \\
 &\Rightarrow \int_a^b F(u)g(u)du = H(b) - H(a) - \int_a^b f(u)G(u)du \\
 &\quad = F(b)G(b) - F(a)G(a) - \int_a^b f(u)G(u)du.
 \end{aligned}$$

Question # show that $f(x) = \sin x$ is Riemann integrable over $[0, \pi/2]$

Sol Take $P = \{0, \frac{\pi}{2n}, \frac{\pi}{n}, \frac{3\pi}{2n}, \dots, \frac{n\pi}{2n}\}$

by dividing $[0, \frac{\pi}{2}]$ into n equal parts

$$\sup f = M_k \quad \text{on } [x_{k-1}, x_k] \quad \Delta x = \frac{\pi}{2n}$$

$$\inf f = m_k \quad \text{on } [x_{k-1}, x_k]$$

$$\text{Then } M_k = \sin \frac{k\pi}{2n} \quad m_k = \sin \frac{(k-1)\pi}{2n}$$

$$U(P, f) - L(P, f) = \sum_{k=1}^n \left(\sin \frac{k\pi}{2n} - \sin \frac{(k-1)\pi}{2n} \right) \cdot \frac{\pi}{2n}$$

$$\leq \frac{\pi}{2n} < \epsilon \quad \text{for } n > \frac{\pi}{2\epsilon} = n_0$$

$\Rightarrow f$ is Riemann integrable over $[0, \frac{\pi}{2}]$

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Mean Value Theorems of Integral

Theorem# If f is continuous on $[a, b]$ and α is increasing on $[a, b]$, then there exists a point c in $[a, b]$ such that

$$\int_a^b f(u) du = f(c) \int_a^b d\alpha = f(c) (\alpha(b) - \alpha(a))$$

Proof# If $\alpha(a) = \alpha(b)$, then theorem is trivially true because both sides are zero and any value of c in $[a, b]$ gives the result.

If f is constant on $[a, b]$ say $f(u) = k$
 If $u \in [a, b]$, then $\int_a^b f d\alpha = k [\alpha(b) - \alpha(a)]$

Hence for any $c \in [a, b]$, we have.

$$\int_a^b f d\alpha = [\alpha(b) - \alpha(a)] f(c)$$

We assume that $\alpha(a) < \alpha(b)$

$\therefore f$ is continuous on $[a, b]$

$\therefore f$ attains its sup & inf on $[a, b]$ & $f \in R(\alpha)$

Let $M = \sup f$ on $[a, b]$

$m = \inf f$ on $[a, b]$

Then

$$m [\alpha(b) - \alpha(a)] \leq \int_a^b f d\alpha \leq M [\alpha(b) - \alpha(a)]$$

$$\Rightarrow m \leq \frac{\int_a^b f d\alpha}{\alpha(b) - \alpha(a)} \leq M$$

If $\frac{\int_a^b f d\alpha}{\alpha(b) - \alpha(a)}$ is equal to either of M, m , then

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theorem is proved because M, m are values of f at some point of $[a, b]$ and that point will be point c required by Theorem.

If $\frac{\int_a^b f d\alpha}{\alpha(b) - \alpha(a)}$ is not equal to M, m , then

$$m < \frac{\int_a^b f d\alpha}{\alpha(b) - \alpha(a)} < M$$

By extreme-value theorem there are points $x_1, x_2 \in [a, b]$ such that

$$f(x_1) = m \quad f(x_2) = M$$

Thus
$$f(x_1) < \frac{1}{\alpha(b) - \alpha(a)} \int_a^b f d\alpha < f(x_2)$$

By intermediate value theorem \exists a point $c \in (x_1, x_2) \subseteq [a, b]$ such that

$$\frac{\int_a^b f d\alpha}{\alpha(b) - \alpha(a)} = f(c)$$

$$\Rightarrow \int_a^b f d\alpha = f(c) [\alpha(b) - \alpha(a)]$$

Theorem # (1st Mean Value Theorem)

If f is continuous on $[a, b]$, $g \in R_\alpha[a, b]$ and g keeps same sign on $[a, b]$, α is increasing on $[a, b]$, then \exists a point $c \in [a, b]$ such that

$$\int_a^b f g d\alpha = f(c) \int_a^b g d\alpha$$

Proof # $\because f$ is continuous on $[a, b]$

$$\therefore f \in R_\alpha[a, b]$$

Also $g \in R_\alpha[a, b]$

$$\Rightarrow fg \in R_a[a, b] \quad \text{III}$$

$$\text{Let } g(u) \geq 0 \quad \forall u \in [a, b]$$

$\therefore f$ is continuous on $[a, b]$

$\therefore f$ attains its sup & inf on $[a, b]$

$$\text{Let } M = \sup f = f(x_2) \quad x_2 \in [a, b]$$

$$m = \inf f = f(x_1) \quad x_1 \in [a, b]$$

$$\text{Then } m \leq f(u) \leq M \quad \forall u \in [a, b]$$

$$\therefore g(u) \geq 0$$

$$\Rightarrow mg(u) \leq f(u)g(u) \leq Mg(u) \quad \forall u \in [a, b]$$

\Rightarrow Integrating

$$m \int_a^b g(u) du \leq \int_a^b f(u)g(u) du \leq M \int_a^b g(u) du$$

$$\Rightarrow m \leq \frac{\int_a^b fg dx}{\int_a^b g(u) dx} \leq M$$

If $\frac{\int_a^b fg dx}{\int_a^b g(u) dx} = M$ or m , then proved. Otherwise

by intermediate value theorem \exists a point $c \in (x_1, x_2) \subseteq [a, b]$ such that

$$\frac{\int_a^b fg dx}{\int_a^b g(u) dx} = f(c)$$

$$\Rightarrow \int_a^b fg dx = f(c) \int_a^b g(u) dx$$

Note Theorem also holds if $g(u) \leq 0 \quad \forall u \in [a, b]$

If we take $g(u) \equiv 1$, then $g \in R_a[a, b]$ & $g \geq 0$

& we have

$$\int_a^b f dx = f(c) [b-a]$$

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Theorem (For Riemann integral) 112

If f is continuous on $[a, b]$, then \exists a point c in $[a, b]$ such that

$$\frac{1}{b-a} \int_a^b f(x) dx = f(c)$$

Proof # If $a=b$ or $f = \text{constant}$, then theorem is trivially proved. Let $a \neq b$, $f \neq \text{constant}$.

$\therefore f$ is continuous

$\therefore f \in R[a, b]$ & attains its sup and inf on $[a, b]$.

Let $x_1, x_2 \in [a, b]$ such that

$$f(x_1) = m = \inf f \quad f(x_2) = M = \sup f$$

Now

$$m(b-a) \leq \int_a^b f dx \leq M(b-a)$$

$$\Rightarrow m(b-a) \leq \int_a^b f dx \leq M(b-a)$$

$$\Rightarrow m \leq \frac{\int_a^b f dx}{b-a} \leq M$$

If $\frac{\int_a^b f dx}{b-a} = M$ or m , then theorem is proved.

Otherwise by intermediate value Theorem \exists a point $c \in (x_1, x_2) \subseteq [a, b]$ such that

$$\frac{\int_a^b f dx}{b-a} = f(c)$$

$$\Rightarrow \int_a^b f dx = f(c)(b-a)$$

Note $\frac{1}{b-a} \int_a^b f dx$ is called mean value of the function on $[a, b]$. This theorem says That a continuous function always assumes its mean value.

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2nd Mean Value Theorem OR Bonnet's Mean Value Theorem

Assume that α is continuous and $\alpha \uparrow$ on $[a, b]$,
 f is monotonically increasing on $[a, b]$, Then there
exists a point $x_0 \in [a, b]$ such that

$$\int_a^b f(x) d\alpha(x) = f(a) \int_a^{x_0} d\alpha(x) + f(b) \int_{x_0}^b d\alpha(x)$$

Proof # $\because \alpha$ is continuous on $[a, b]$

$\therefore \alpha$ is in $R_f[a, b]$

Also $f \in R_\alpha[a, b]$ and by parts integration
we have

$$\int_a^b f(x) d\alpha(x) = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha(x) df(x) \rightarrow \textcircled{1}$$

Now in integral $\int_a^b \alpha(x) df$, α is continuous
and f is \uparrow on $[a, b]$. Therefore by above mean
value theorem there exists a point $x_0 \in [a, b]$
such that

$$\int_a^b \alpha(x) df = \alpha(x_0) [f(b) - f(a)]$$

using in $\textcircled{1}$

$$\begin{aligned} \int_a^b f d\alpha &= f(b)\alpha(b) - f(a)\alpha(a) - \alpha(x_0)f(b) + \alpha(x_0)f(a) \\ &= f(a) [\alpha(x_0) - \alpha(a)] + f(b) [\alpha(b) - \alpha(x_0)] \\ &= f(a) \int_a^{x_0} d\alpha(x) + f(b) \int_{x_0}^b d\alpha(x) \end{aligned}$$

Corollary # If f is continuous and monotone increasing on $[a, b]$, there exists a point c in $[a, b]$, such that

$$\int_a^b f(t) dt = f(a)(c-a) + f(b)(b-c)$$

Proof # In above theorem take $f(t)$ and $\alpha(t) = t$ in place of α , then for $x_0 = c$

$$\int_a^b f(t) dt = f(a)(c-a) + f(b)(b-c)$$

Theorem # (2nd MVT for Riemann Integration)

If f is continuous and increasing on $[a, b]$ and g is non-negative and integrable on $[a, b]$, then \exists a point $x_0 \in [a, b]$ such that

$$\int_a^b fg dx = f(a) \int_a^{x_0} g dx + f(b) \int_{x_0}^b g dx.$$

Proof # By 1st Mean value theorem \exists a point $y \in [a, b]$ such that

$$\int_a^b fg dx = f(y) \int_a^b g dx \quad \longrightarrow \textcircled{1}$$

Also since f is increasing, therefore

$$f(a) \leq f(y) \leq f(b)$$

Consider the function G defined on $[a, b]$ by

$$G(u) = [f(b) - f(a)] \int_u^b g dx$$

$\because g$ is non-negative, G is a decreasing function.

Further

$$G(a) = [f(b) - f(a)] \int_a^b g dx$$

$$\geq [f(y) - f(a)] \int_a^b g dx$$

$$\geq 0 = G(b) \quad \Rightarrow G(a) \geq G(b)$$

$\therefore G$ is continuous on $[a, b]$
 \therefore By intermediate value theorem \exists a point
 $x_0 \in [a, b]$ such that

$$G(x_0) = [f(b) - f(a)] \int_a^b g dx$$

Now

$$G(x_0) = [f(b) - f(a)] \int_{x_0}^b g dx = [f(b) - f(a)] \int_a^b g dx$$

we have

$$\begin{aligned}
 f(b) \int_{x_0}^b g dx - f(a) \int_{x_0}^b g dx &= f(b) \int_a^b g dx - f(a) \int_a^b g dx \\
 &= \int_a^b f(b) g(x) dx - \int_a^b f(a) g(x) dx \\
 \Rightarrow \int_a^b f g dx &= f(a) \left[\int_a^b g dx - \int_{x_0}^b g dx \right] + f(b) \int_{x_0}^b g dx \\
 &= f(a) \int_a^{x_0} g dx + f(b) \int_{x_0}^b g dx
 \end{aligned}$$

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Question # prove that $\frac{\pi^2}{9} \leq \int_{\pi/6}^{\pi/2} \frac{x}{\sin x} dx \leq \frac{2\pi^2}{9}$

Sol let $f(x) = \frac{1}{\sin x}$ & $g(x) = x$

Then f, g are continuous on $[\pi/6, \pi/2]$ & hence integrable on $[\pi/6, \pi/2]$

Also $g(x) = x > 0 \quad \forall x \in [\pi/6, \pi/2]$

Since f decreasing on $[\pi/6, \pi/2]$

$$\inf f = f(\pi/2) = 1 \quad \sup f = f(\pi/6) = 2$$

By 1st Mean value Theorem \exists a point $c \in [\pi/6, \pi/2]$ such that

$$\begin{aligned}
 \int_{\pi/6}^{\pi/2} f(x) g(x) dx &= f(c) \int_{\pi/6}^{\pi/2} g(x) dx \\
 \Rightarrow \int_{\pi/6}^{\pi/2} \frac{x}{\sin x} dx &= f(c) \int_{\pi/6}^{\pi/2} x dx = f(c) \cdot \frac{\pi^2}{9}
 \end{aligned}$$

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$$\Rightarrow \int_{\pi/6}^{\pi/2} \frac{x}{\sin x} dx = f(c) \cdot \frac{\pi^2}{9}$$

$\therefore f$ is continuous on $[\pi/6, \pi/2]$ & $c \in [\pi/6, \pi/2]$

$$\therefore \pi/6 \leq c \leq \pi/2$$

$$\Rightarrow f(\pi/6) \leq f(c) \leq f(\pi/2)$$

$$\Rightarrow 1 \leq f(c) \leq 2$$

$$\Rightarrow 1 \leq \frac{9}{\pi^2} \int_{\pi/6}^{\pi/2} \frac{x}{\sin x} dx \leq 2$$

$$\Rightarrow \frac{\pi^2}{9} \leq \int_{\pi/6}^{\pi/2} \frac{x}{\sin x} dx \leq \frac{2\pi^2}{9}$$

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